WRMR Analysis: Choice of Optimal Wavelet Families for the Adaptive Solution of Nonlinear PDEs

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What are Wavelets Commonly Used For and Why?

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- Signal processing: Flexible, efficient signal representation and decomposition: Beat FT, WFT, CT, DCT, etc. The High Road in DSP ("A Wavelet Tour of Signal Processing" by Stephane Mallat)
- Data Compression: FBI fingerprint archives 26:1 wavelet based compression, otherwise, at 500 pixels/inch and 256 levels of gray-scale information per pixel, one crook = 6MBytes, entire FBI database = 200 Terabytes (30 Mega-suspects) @ $1000/Gbyte = $ 200 Megabucks! Sparse representations: Average data (smooth, well represented) + details (successively ignored) => Subband Coding
- Denoising: Recovering Brahms himself playing Hungarian dance number 1 in 1889. Hear it @ http://www.music.yale.edu Shrinkage and Thresholding: Keep sharp features, lose the noise. See what the experts have to say: **http://www-stat.stanford.edu/~donoho/Reports/**
- Pattern Detection, self similarity, coherent structures: See El Nino's regularity for yourself in Chi ^2 distribution of wavelet power => time series had Gaussian statistics: http://paos.colorado.edu/research/wavelets/wavelet1.html

What Are Wavelets? Start @ (www.wavelets.org) & Surf (Mathsoft, amara, ...)

Mallat, Meyer, Daubechies, Beylkin, Coifman, Strang, Sweldens, Donoho...

- Wavelets are localized kernels or atoms in PHASE SPACE.
- You may think of them as basis functions with prescribed dilation and translation properties.
- They may or may not be orthonormal or have compact support or be differentiable everywhere, or be fractal, or have many zero momemts.
- Wavelets are like breathing wave packets which can home in on structures in phase space better than FT or WFT ever could.

$$\psi_{j,k}(x) = 2^{j/2}\,\psi\left[2^j\left(x - \frac{k}{2^j}\right)\right] \quad j,k \in Z$$

$$\Psi_n(x) = (-1)^n \frac{d^n}{dx^n}\left[\exp\left(-\kappa\,(x - x_c)^2/2\right)\right]$$

When the scale is decreased translation steps between wavelets should likewise be decreased

What is MRD or Multi-resolution Decomposition?

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- Multiresolution: Zoom in and out on a number of successively finer scales in a sequence of nested approximation subspaces $\{V_j\}_{j \text{ in } Z}$.
- In general, get an overcomplete basis set in $L_2(R)$. Approximate (or truncate) by bounding the scales of interest.

Scaling functions and the scaling equation:

Low pass filter

$$\varphi(x) = 2\sum_{k=0}^{2N-1} h_k \varphi(2x-k)$$

$$\sum_k h_k = 1 \qquad \int_{-\infty}^{\infty} \varphi(x)\,dx = 1$$

The Wavelets:

High pass filter

$$\psi(x) = 2\sum_{k=0}^{2N-1} g_k \varphi(2x-k)$$

$$g_k = (-1)^k h_{2N-1-k}$$

These filters decompose a sampled signal into 2 sub-sampled channels: the coarse approximation of the signal and the missing details at finer scales. The original signal can be reconstructed from these channels by interpolation.

Discrete Wavelet Transforms & Perfect Reconstruction Subband Coding Filters

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DWTs are Orthonrmal decompositions:

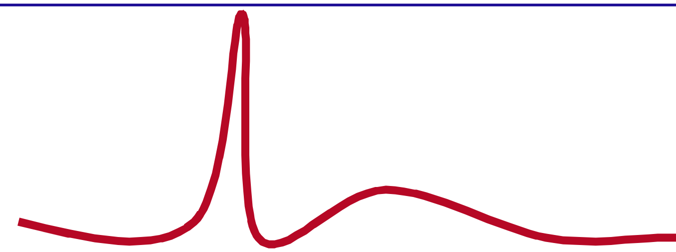
$$f(t) = \sum_k c_k \phi_k(t) + \sum_{j=0}^{J} \sum_k d_{jk} \psi_{jk}(t)$$

$$c_m = \int f(t)\,\phi_m(t)\,dt, \quad d_{lm} = \int f(t)\,\psi_{lm}(t)\,dt$$

The number of operations required to perform DWTs with a filter of length L (with L taps) is of order L x N (even FFTs require N ln N operations)

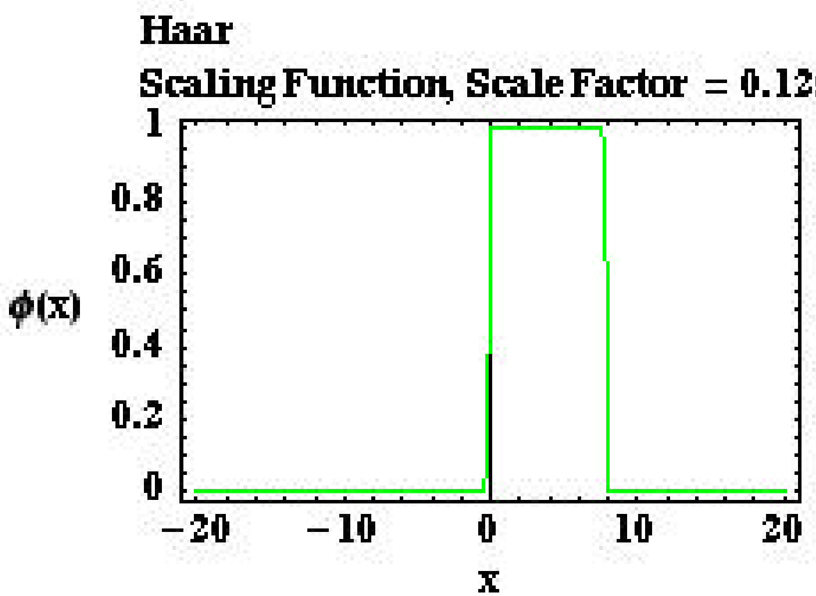
$$L\,N\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right) < 2LN$$

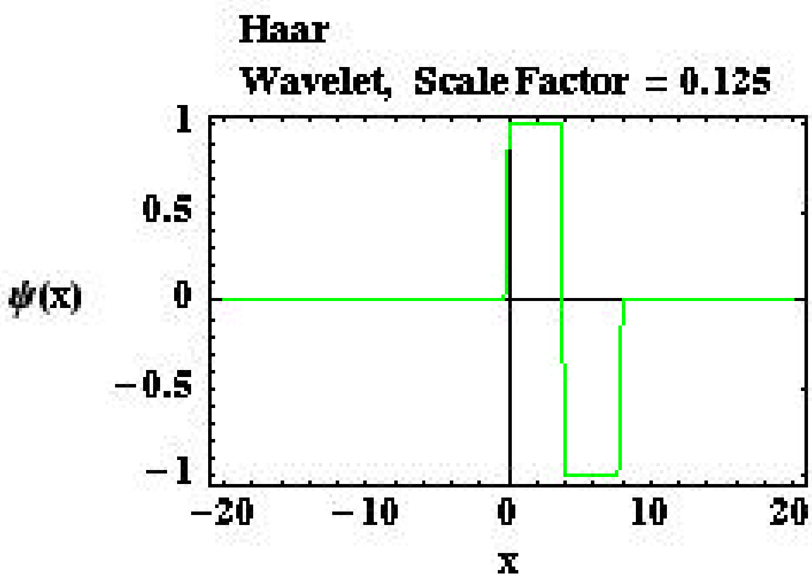
The Key to Multi-Rresolution Analysis Using Wavelets Is:



- **THRESHOLDING**
- **Two Ways to do it:**
- **Linear or Largest Scale Thresholding**
- **Nonlinear or Largest Coefficient Thresholding**
- **Linear is Fourier like: Keep up to some scale and chop off the rest**
- **Nonlinear Thresholding is the true breakthrough: Keep those wavelets which have the largest coefficients no matter where they are and on whatever scale they are. No need to keep intermediate scales or intermediate locations. Just keep the BIG ones. Automatically denoise, automatically compress and automatically bring out significant patterns.**

The Scaling Function and Wavelet for Haar or Daubechies 1 in X-Space





The Scaling Function and Wavelet for Haar or Daubechies 1 in K- Space

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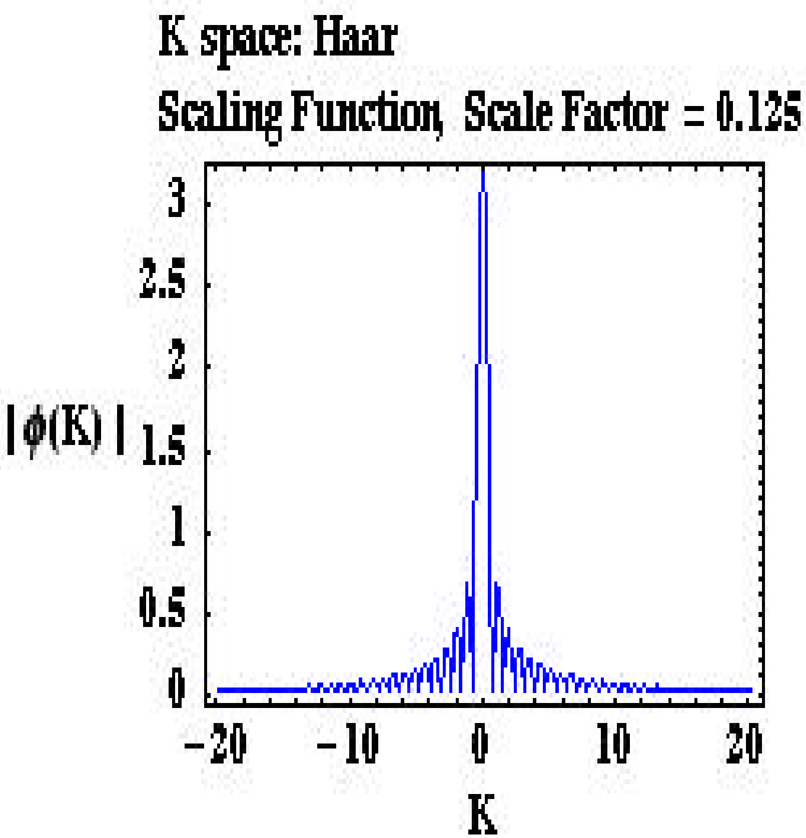

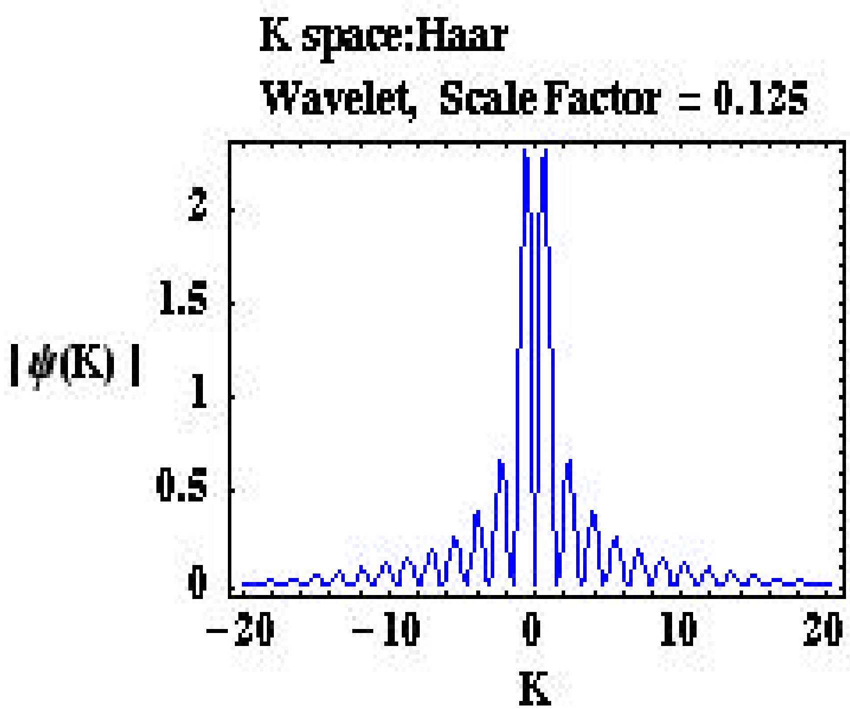

The Scaling Functions and Wavelets for Daubechies 2-6 in X-Space

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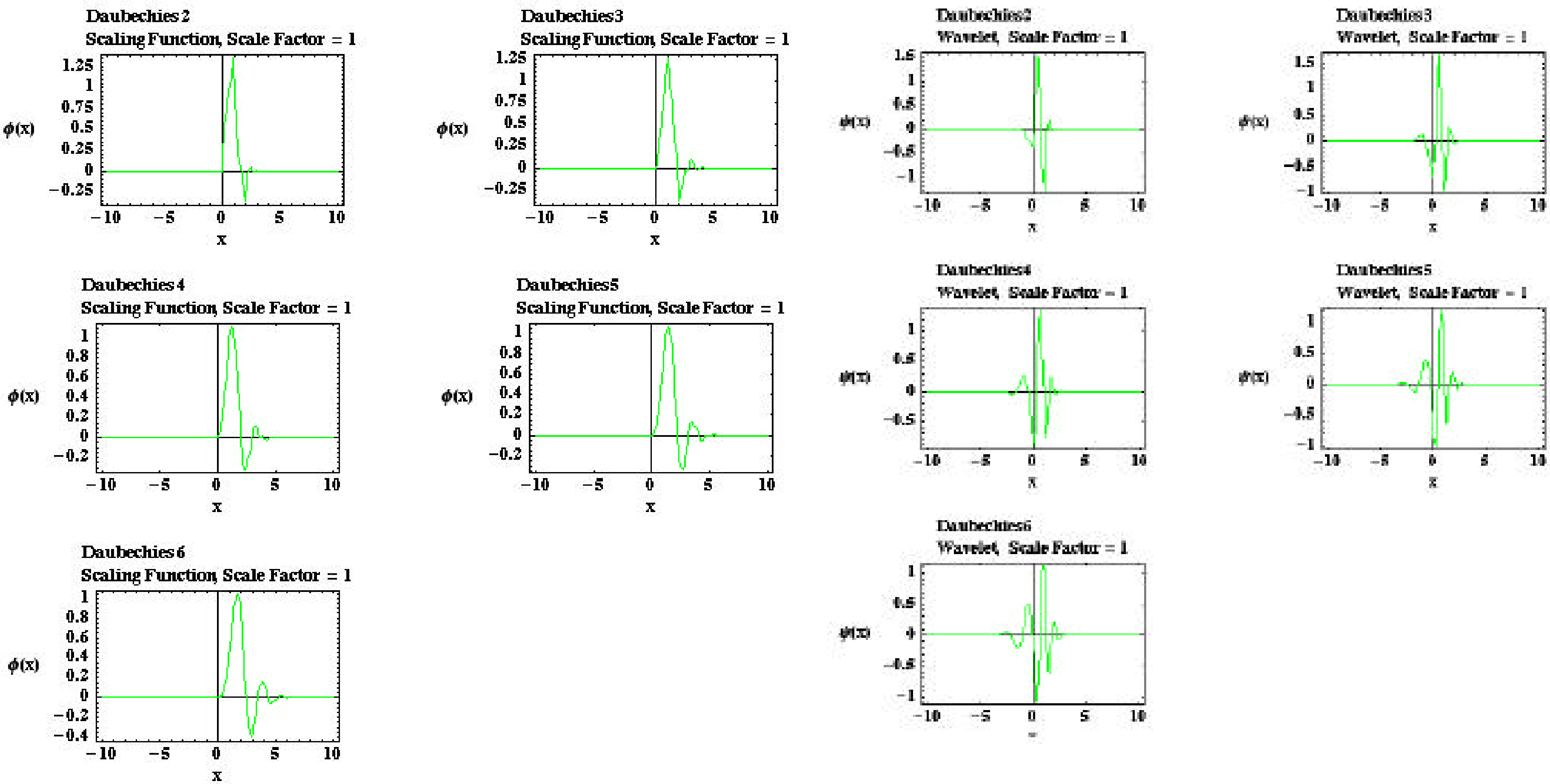
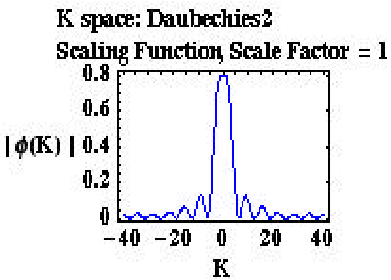
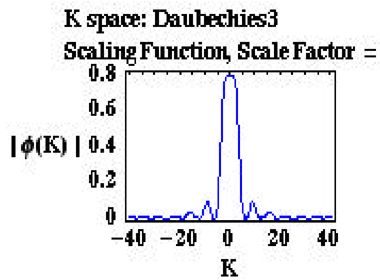
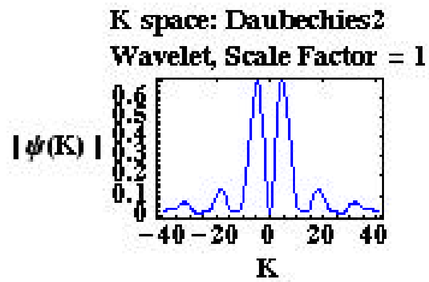
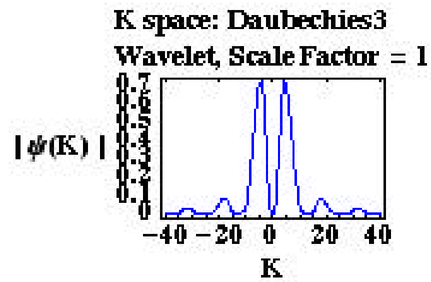


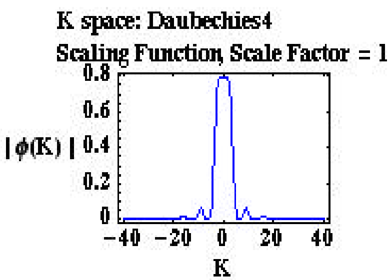
The Scaling Functions and Wavelets for Daubechies 2-6 in k-Space

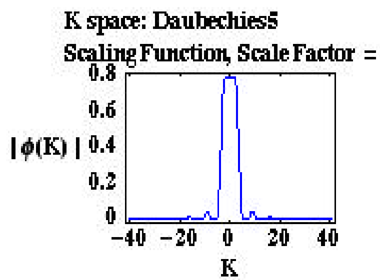


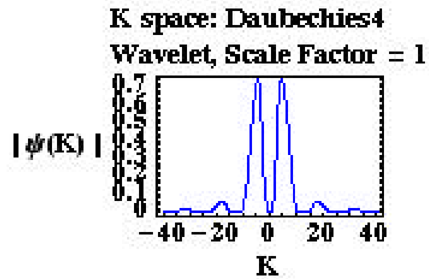


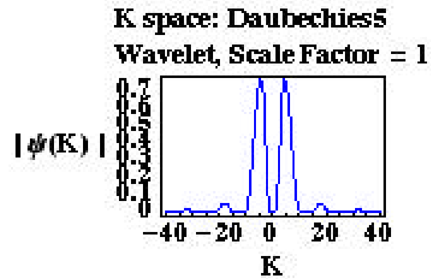


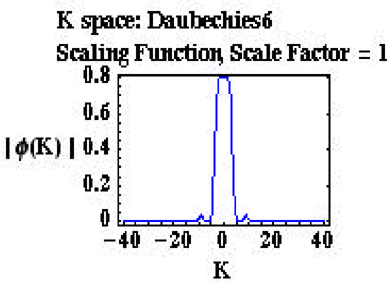


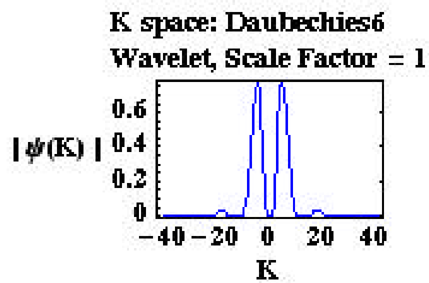












Outline

- Wavelets and PDEs in Non Linear Plasma Physics and Photonics
- A Toy Problem: A band limited signal (over)analyzed
- Wavelet (DWT) Analysis of the Solution of Burger's Equation with Random Driving. MRD and its implementation in a mathematica notebook
- Wigner function representations of the MRD
- WRMR measure and optimum wavelet selection algorithms applied to random shocks (or solitons, or...)

The Big Picture

- Original motivation: Want to implement adaptive wavelet based schemes for solving NL evolution (eg. parabolic) PDEs
- How should one choose which WLT family to use?
- Traditional measures do not discriminate very adequately
- Want to amplify the differences between potential choices making the optimal choice automatable. How?
- Use the Wigner function representation of the MRD: WRMR analysis

An AI Project for PDEs: Original Raison d'Etre of this Work

- **Ambition: Use adaptive wavelet techniques to help a PDE teach itself how to solve itself.**
- **How? Look at optimum compact and sufficiently accurate representations of the solution of an evolutionary eq. such as a parabolic PDE, for instance, over a library of wavelets and representations. Find the optimum sequence of grids or coefficients which can sequentially represent the solution very well.**
- **Do all this work for one set of ICs and BCs, parameters and coefficients and *then* while running some nearby problem try using the same sequence of wavelets as found to be appropriate in previous cases.**
- **Most numerical problems need not have optimum single case runs but be optimum or efficient over a sequence of 100 or 1000 runs. This is where adaptive wavelet techniques could score big, one hopes!**

The Generalized Burger's-KdV Equation & It's Split Operator Solution Using Adaptive Wavelets

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$$\frac{\partial u}{\partial t} + A u^{\alpha} \frac{\partial u}{\partial x} + B \frac{\partial^{2\beta+1} u}{\partial x^{2\beta+1}} - C \frac{\partial^{2\gamma} u}{\partial x^{2\gamma}} = \frac{\partial S}{\partial x}$$

Nonlinear Steepening: Shocks, Advection

Dispersion: Wave Like Phenomena

Dissipation: Diffusion

Source Term: External Driver RandomSpace-Time Dependent "Burgerlence"

Solve along Characteristics

$$\frac{\partial u^{(1)}}{\partial t} + A u^{(1)\alpha} \frac{\partial u^{(1)}}{\partial x} = \frac{\partial S}{\partial x}$$

Interpolate onto Wavelet Thresholding Determined Nonuniform Adaptive Grid

Solve Using Wavelets (Collocation) or Finite Differencing on Non-uniform Grid

$$\frac{\partial u^{(2)}}{\partial t} + B \frac{\partial^{2\beta+1} u^{(2)}}{\partial x^{2\beta+1}} - C \frac{\partial^{2\gamma} u^{(2)}}{\partial x^{2\gamma}} = 0$$

A Damped Driven Inhomogeneous Nonlinear Schrodinger Equation Models Chaotic Interactions Between Highly Localized, Resonantly Generated Solitons

$$i\frac{\partial A}{\partial \tau} - \frac{\partial^2 A}{\partial \varsigma^2} + \left[\varsigma - i\Gamma_e - p|A|^2\right]A = A_0(\tau)$$

$$\Gamma_e = \frac{\nu_e}{\omega_0}(k_D L)^{2/3}$$

Damping Coefficient

$$p = \left(\frac{v_{osc,MAX}^2}{v_{th}^2}\right)(k_D L)^2$$

Nonlinearity Coefficient

$$A_0 = \frac{B_y(0,\tau)\sin\theta}{\left[B_y(0,\tau)\sin\theta\right]_{MAX}}$$

Driver term

$$\varsigma = \frac{z}{L}(k_D L)^{2/3}$$

$$\tau = \frac{\omega_0 t}{2}\frac{1}{(k_D L)^{2/3}}$$

$$A = \frac{\tilde{E}_z}{\left[B_y(0,t)\sin\theta\right]_{MAX}}\left[\frac{1}{(k_D L)^{2/3}}\right]$$

Three Special Limits of the Generalized NLS Equation Describing Res. Abs.

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$$i\frac{\partial A}{\partial \tau} - \frac{\partial^2 A}{\partial \varsigma^2} + \left[\varsigma - i\Gamma_e\right]A = A_0$$

Exactly solvable Linear PDE via Laplace transforms and in terms of driven Airy functions (messy)

$$i\frac{\partial A}{\partial \tau} - \frac{\partial^2 A}{\partial \varsigma^2} - p|A|^2 A = 0$$

Exactly solvable nonlinear PDE in terms of solitons (elegant)

$$\frac{\partial^2 A}{\partial \varsigma^2} + \left[\varsigma - i\Gamma_e - p|A|^2\right]A = A_0$$

A nonlinear ODE for steady state solutions of the driven complex eigenvalue problem. Requires a BVP solver. Can you get there from here?

Propagation in Nonlinear Waveguides: Saturable Gain/Abs., Saturable NL, Complex Index Structure (PBG)

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$$
\begin{aligned}
-i\frac{\partial \mathbf{E}}{\partial \tau} = \; & {}^{2}\mathbf{E} + \bar{n}_0^2(\mathbf{x})\left[1 + n_2\, h_{NL}(\mathbf{x})\left(1 - \exp\left[-|\mathbf{E}|^2/|\mathbf{E}_{NLS}|^2\right]\right)\right]\mathbf{E} \\
& + i\,\bar{n}_0^2(\mathbf{x})\, v(\mathbf{x})\mathbf{E} \\
& - i\,\bar{n}_0^2(\mathbf{x})\left[\frac{\gamma_2 h_G(\mathbf{x})}{1 + |\mathbf{E}|^2/|\mathbf{E}_{GS}|^2} - \frac{v_2 h_A(\mathbf{x})}{1 + |\mathbf{E}|^2/|\mathbf{E}_{AS}|^2}\right]\mathbf{E} \\
& + \hat{S}(\mathbf{x})
\end{aligned}
$$

Use a split step scheme with the ^{2} operator being inverted using a nonuniform FFT or fully Wavelet based scheme where the appropriate nonuniform grid is adaptive and found by largest coefficient thresholding of the DWT of the solution in the previous time step.

Sample Coefficients of the NL Waveguide Propagation Equation

$$\bar{n}_0^2(x,z)=\left(\frac{n_{0,\text{inner}}^2}{n_{0,\text{outer}}^2}\right)\sum_{l=1}^{N}\frac{\alpha_l}{\sqrt{N}}\,e^{-\left[\left(z-z_{c_l}\right)^2/z_{w_l}^2+\left(x-x_{c_l}\right)^2/x_{w_l}^2\right]}\cos\left[k_{z_l}\left(z-z_{c_l}\right)+k_{x_l}\left(x-x_{c_l}\right)+\phi_l\right]$$

$$\times\, h_{\text{taper}}(x,z)$$

$$\nu(z)=\frac{\nu_l}{2}h_l(z)+\frac{\nu_r}{2}h_r(z)$$

$$h_{l,r}(z)=\left\{1\mp\tanh\left[\alpha_{l,r}\left(z-z_{l,r}\right)\right]\right\}$$

$$h_{\text{taper}}(x,z)=\text{sech}\left[\frac{\alpha\, x}{z-z_{\text{taper}}}\right]$$

$$\hat{S}(x,z)=\hat{s}\,e^{i\left(k_{z_S}z+k_{x_S}x\right)}\,e^{-\left[\left(z-z_{c_S}\right)^2/z_{w_S}^2+\left(x-x_{c_S}\right)^2/x_{w_S}^2\right]}$$

Split Step (A) of the 3 Split Step Decomposition of the NL Waveguide Propagation Equation

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$$-i\frac{\partial \mathbf{E}^{(1)}}{\partial \tau} = {}^{2}\mathbf{E}^{(1)} + \hat{\mathbf{S}}(x,z)$$

$$\mathbf{E}^{(1)}(\tau,\mathbf{k}) = \mathbf{E}^{(1)}_{init}(\tau_{init},\mathbf{k})e^{\left[-ik^2-2\nu(k)\right]\Delta\tau} - \frac{\hat{\mathbf{S}}(\mathbf{k})\left[1-e^{\left[-ik^2-2\nu(k)\right]\Delta\tau}\right]}{\left[k^2-2i\nu(k)\right]}$$

$$\mathbf{E}^{(1)}(\tau,\mathbf{x}) = \mathrm{FFT}^{-1}\left\{\mathbf{E}^{(1)}(\tau,\mathbf{k})\right\}$$

Split Step (B) of the 3 Split Step Decomposition of the NL Waveguide Propagation Equation

$$-i\frac{\partial \mathbf{E}^{(2)}}{\partial \tau} = \bar{n}_0^2(\mathbf{x})\left[1 + n_2\, h_{NL}(\mathbf{x})\left(1 - e^{-|\mathbf{E}|^2/|\mathbf{E}_{\mathrm{NLS}}|^2}\right) + i\nu(\mathbf{x})\right]\mathbf{E}^{(2)}$$

$$\mathbf{E}^{(2)}(\tau,\mathbf{x}) = \mathbf{E}^{(2)}_{init}(\tau_{init},\mathbf{x}) \times \exp\left\{ i\,\bar{n}_0^2(\mathbf{x})\left[1 + n_2\, h_{NL}(\mathbf{x})\left(1 - e^{-|\mathbf{E}^{(2)}_{init}|^2/|\mathbf{E}_{\mathrm{NLS}}|^2}\right) + i\nu(\mathbf{x})\right]\Delta\tau\right\}$$

Split Step (C) of the 3 Split Step Decomposition of the NL Waveguide Propagation Equation

$$-i\frac{\partial \mathbf{E}^{(3)}}{\partial \tau} = -i\,\bar{n}_0^2(\mathbf{x})\left[\frac{\gamma_2 h_G(\mathbf{x})}{1+|\mathbf{E}|^2/|\mathbf{E}_{GS}|^2} - \frac{\nu_2 h_A(\mathbf{x})}{1+|\mathbf{E}|^2/|\mathbf{E}_{AS}|^2}\right]\mathbf{E}^{(3)}$$

$$X_A = |\mathrm{E}_{AS}|^2;\; X_G = |\mathrm{E}_{GS}|^2$$

$$\mathbf{E}^{(3)}(\tau) = \mathbf{E}^{(3)}(\tau_0)\times\sqrt{\frac{X}{X_{init}}}$$

$$\bar{X} = \frac{(1-\nu_2/\gamma_2)|\mathrm{E}_{AS}|^2}{\left[1-(\nu_2/\gamma_2)\left(|\mathrm{E}_{AS}|^2/|\mathrm{E}_{GS}|^2\right)\right]}$$

$$\left[\frac{X/X_{init}}{\left((X+\bar{X})/(X_{init}+\bar{X})\right)}\right]^{X_A/\bar{X}} \exp\left[(X-X_{init})/X_G\right]\times$$

$$\left((X+\bar{X})/(X_{init}+\bar{X})\right)^{(X_A+X_G-\bar{X})/X_G} = e^{2\gamma_2 \bar{n}_0^2(\mathbf{x})h_G(\mathbf{x})\left[1-(\nu_2/\gamma_2)\left(|\mathrm{E}_{AS}|^2/|\mathrm{E}_{GS}|^2\right)\right]\Delta\tau}$$

How About Kinetic Theory and AWCT? Tackle the Vlasov Eqn. in Double Phase Space (x, k, v, v_k)

Solve the IVP for the Vlasov-Poisson system of equations:

$$\frac{\partial f_e}{\partial t} + \mathrm{v}\frac{\partial f_e}{\partial x} - \frac{q\,E(x,t)}{m}\frac{\partial f_e}{\partial \mathrm{v}} = 0$$

$$\frac{\partial E(x,t)}{\partial x} = 4\pi q\left(1 - \int_{-\infty}^{\infty} f_e(x,\mathrm{v},\mathrm{t})\, d\mathrm{v}\right)$$

Compare the results to a semi-Lagrangian Nancy Vlasov Code (P. Bertrand)

Add FP term to control HOTs in the Hermite like expansion:

$$\left(\frac{\partial f_e}{\partial t}\right)_{coll} = \nu\frac{\partial}{\partial \mathrm{v}}\left\{\mathrm{v}\, f_e + \mathrm{v}_0^2\frac{\partial f_e}{\partial \mathrm{v}}\right\}$$

$$E(x,t) = \sum_{i=1}^{N_x} \alpha_i(t)\,\Phi_{E,i}(x)$$

Tile Phase Space in Optimum Wavelet-Composed Non-uniform Patches

$$f_e(x,\mathrm{v},\mathrm{t}) = \sum_{j=1}^{N_x}\sum_{k=1}^{N_\mathrm{v}} \beta_{\mathrm{j,k}}(t)\,\Phi_{f_e,j}(x)\,\Psi_{f_e,k}(\mathrm{v})$$

$$1 < \nu\, N_\mathrm{v} < 10$$

SRS & STEAS Mimicking, Ponderomotive Force Driven, Vlasov-Poisson System of Equations

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Vlasov

$$\frac{\partial f_e^{1D}}{\partial \bar{t}} + \bar{v}\frac{\partial f_e^{1D}}{\partial \bar{z}} - \left(E - \frac{\partial \psi_{PF}}{\partial \bar{z}}\right)\frac{\partial f_e^{1D}}{\partial \bar{v}} = 0$$

$$\bar{t} = \omega_{pe} t;\ \bar{z} = z/\lambda_{De};\ \bar{v} = v/v_{th}$$

$$\frac{\partial E}{\partial \bar{z}} = 1 - \int f_e^{1D}\, d\bar{v}$$

Poisson

$$\int v^2 f_e^{3D}\, dv^3 = 3\, v_{th}^2$$

$$\psi_{PF} = \sum_{\#driver\ modes} \psi_{AMP}^{(i)} \cos\left(\bar{k}_i \bar{z} - \bar{\omega}_i \bar{t}\right)$$

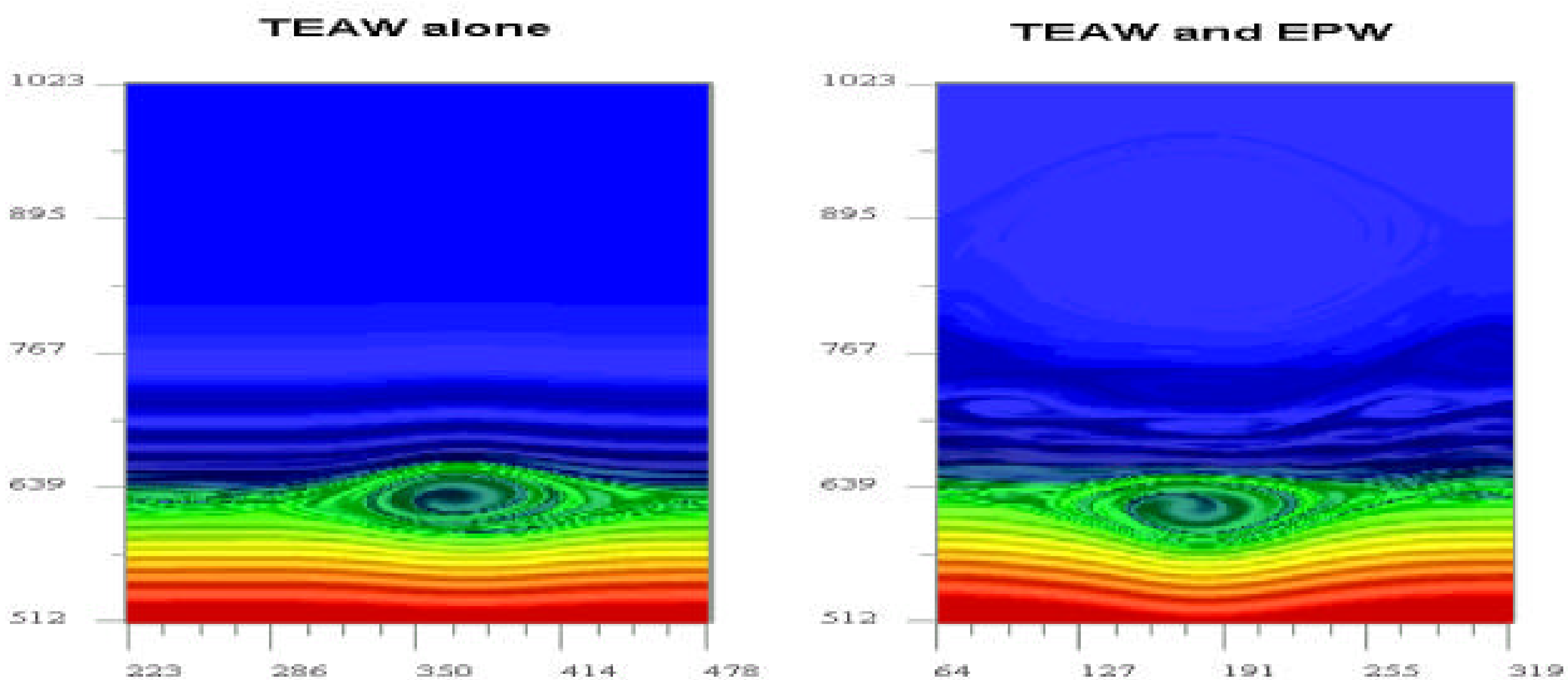
$$\psi_{AMP} = \frac{\left(\frac{eE_0}{m\omega_0}\right)\left(\frac{eE_s}{m\omega_s}\right)}{v_{th}^2}$$

$$\psi_{AMP} = \frac{0.037}{T_{e,keV}}\left(I_{0,10^{14}\,W/cm^2}\lambda_{0\,\mu m}^2\right)\sqrt{\frac{I_s}{I_0}\left(\frac{\lambda_s}{\lambda_0}\right)}$$

$$\frac{\partial \psi_{PF}}{\partial \bar{z}} = -\sum_{\#\,driver\ modes} \psi_{AMP}^{(i)}\, \bar{k}_i \sin\left(\bar{k}_i \bar{z} - \bar{\omega}_i \bar{t}\right)$$

Capturing the Interaction Between Driven and Released EPW & TEAW $k\lambda_D = 0.26$, $\omega_{TEAW}:\omega_{EPW} = 1:3$

The gradual invasion of the TEAW space by the evolution of a driven and released EPW is shown in this snapshot comparing the phase spaces of TEAW formation without and then with a pe-existent EPW. **TEAW drive amplitude is at 0.03 while the EPW's is 0.003.**



EPW and TEAW Coexistence & Interaction Are Strongly Affected by the Initial e⁻ VDF

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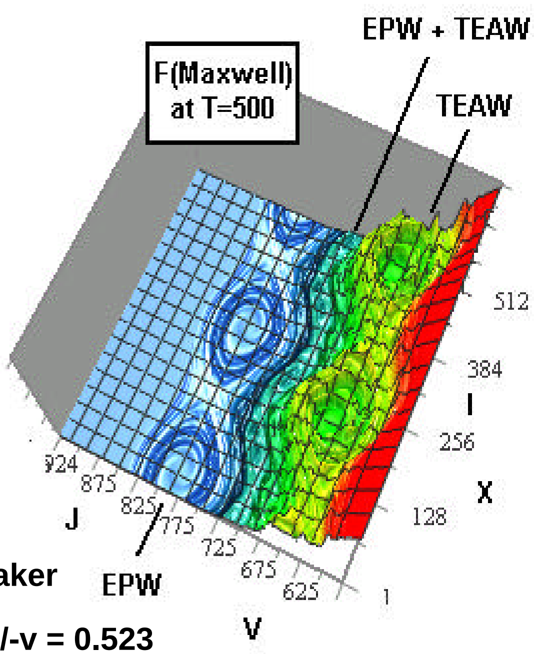
At high K (=0.3927 = π / 8) the distribution function shape makes a big difference to the EPW. It is presumably this that affects the interacting TEAW.

TEAW is stronger +/- v = 0.523 (vphase = 1.54)

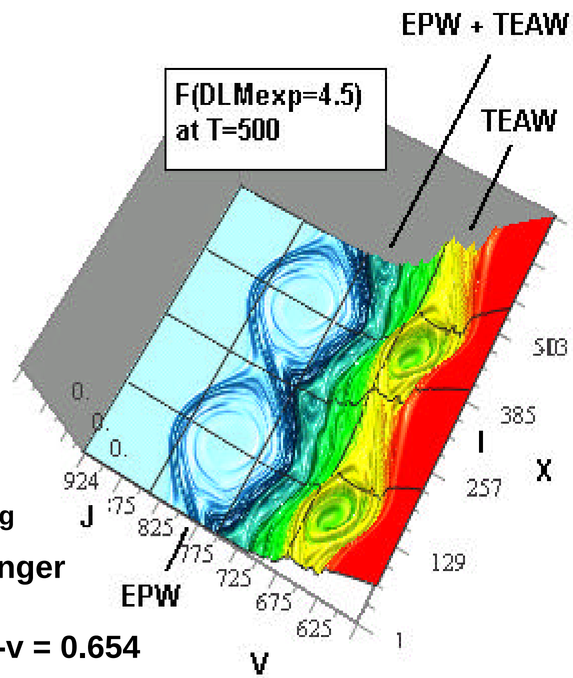
TEAW is weaker +/- v = 0.351



More damping

weaker

+/-v = 0.523



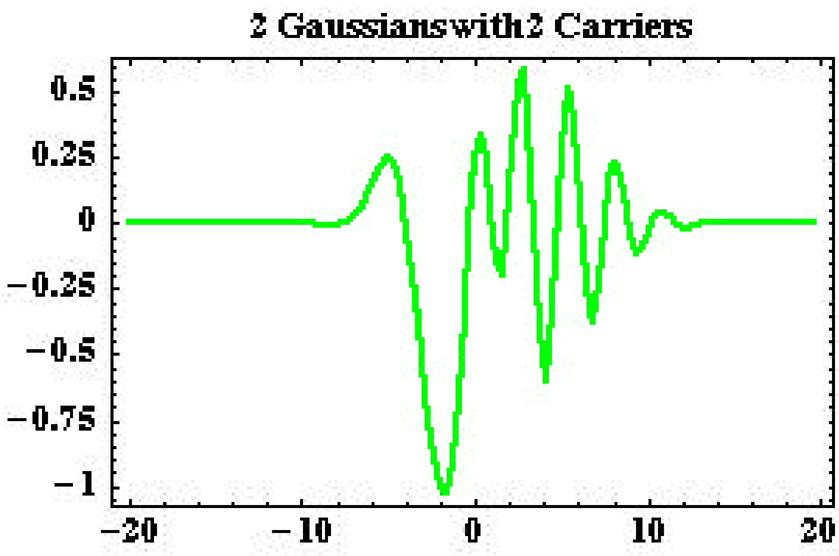
Less damping

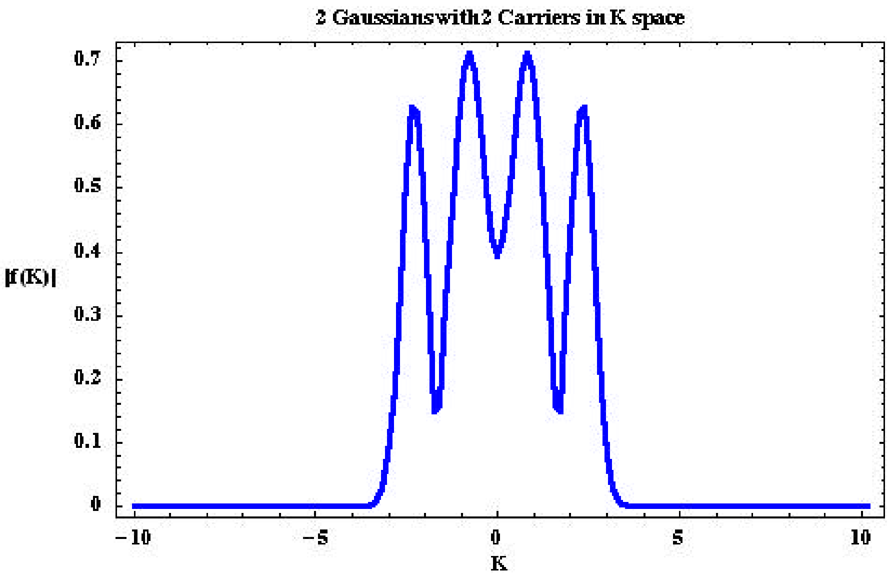
stronger

+/-v = 0.654

A Toy Problem: A Double Gaussian Wavepacket with Two Carriers

$$f(x) = \exp\left[-(x+2)^2/8\right] \times \sin[0.8\,x] + 0.6 \times \exp\left[-(x-4)^2/18\right] \times \sin[2.3\,x + \pi/2]$$





The Largest Coefficients' Amplitudes in Descending Order

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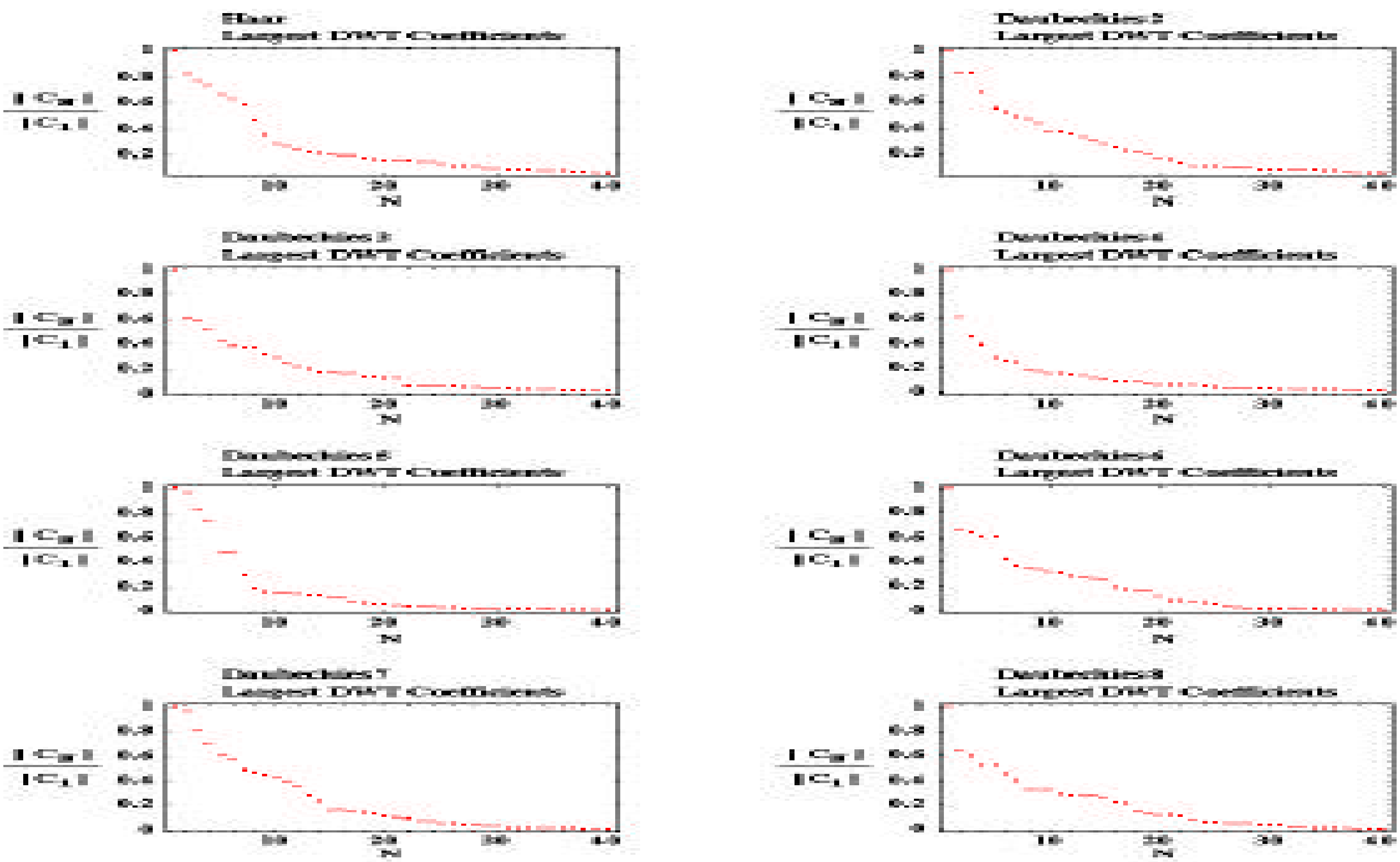
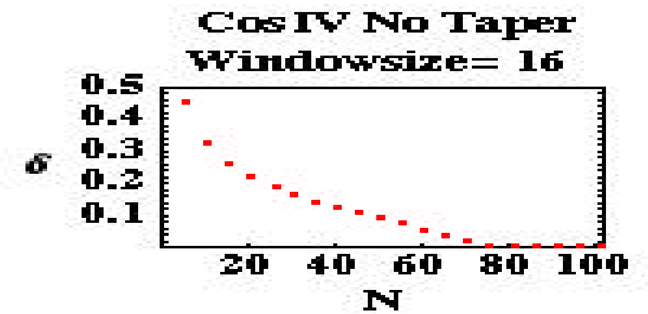
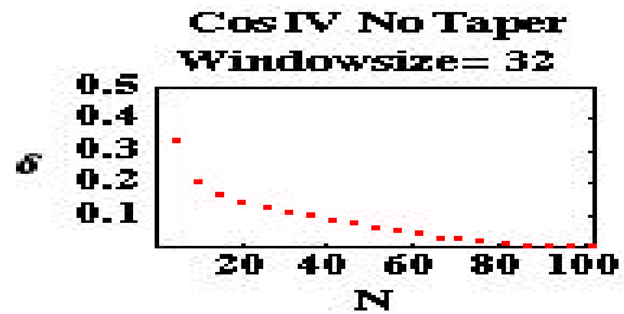
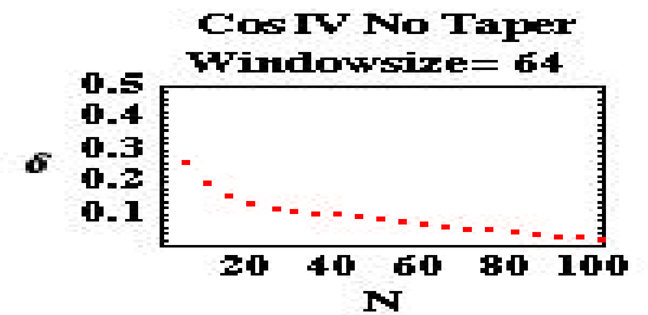
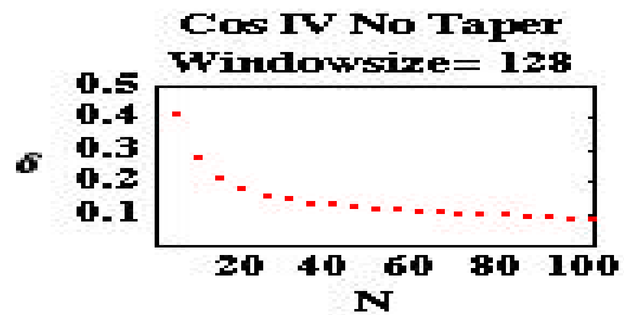


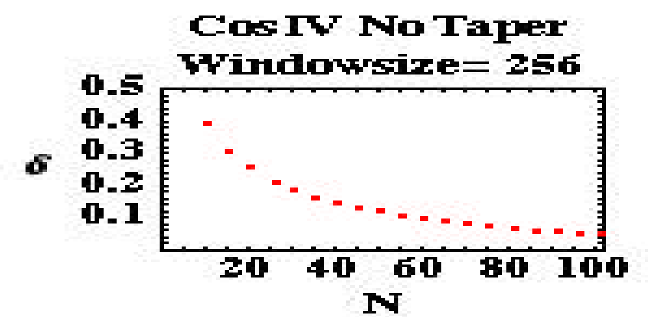
How Well Do Box Windowed Fourier Transforms Do?











How Well Do Tapered Windowed Fourier Transforms Do?

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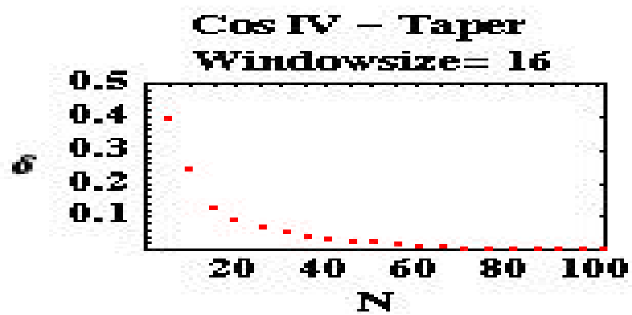
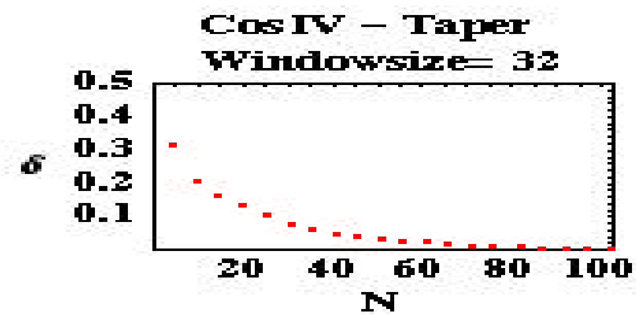
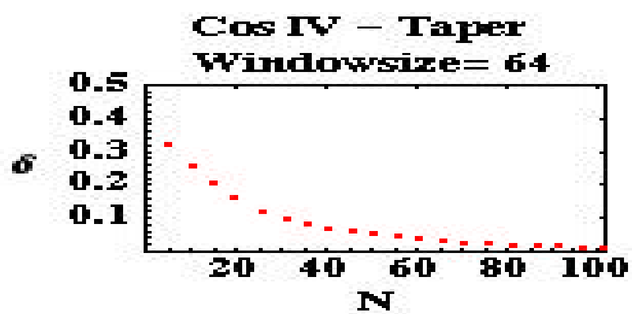
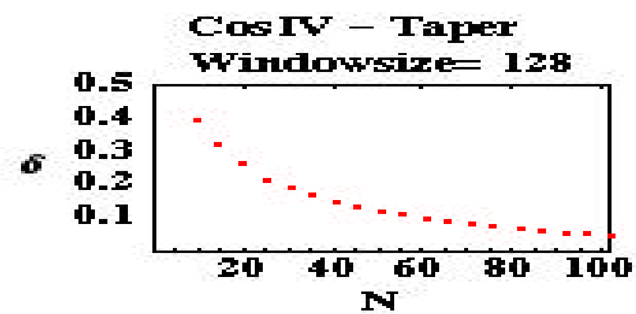
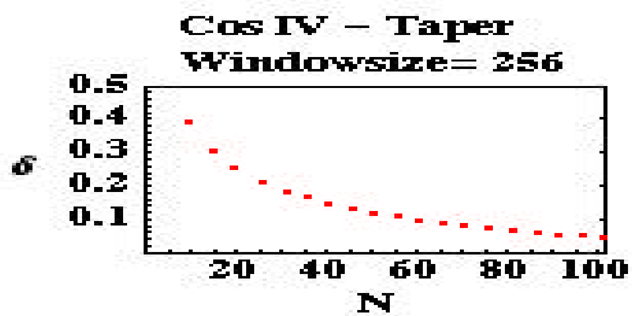


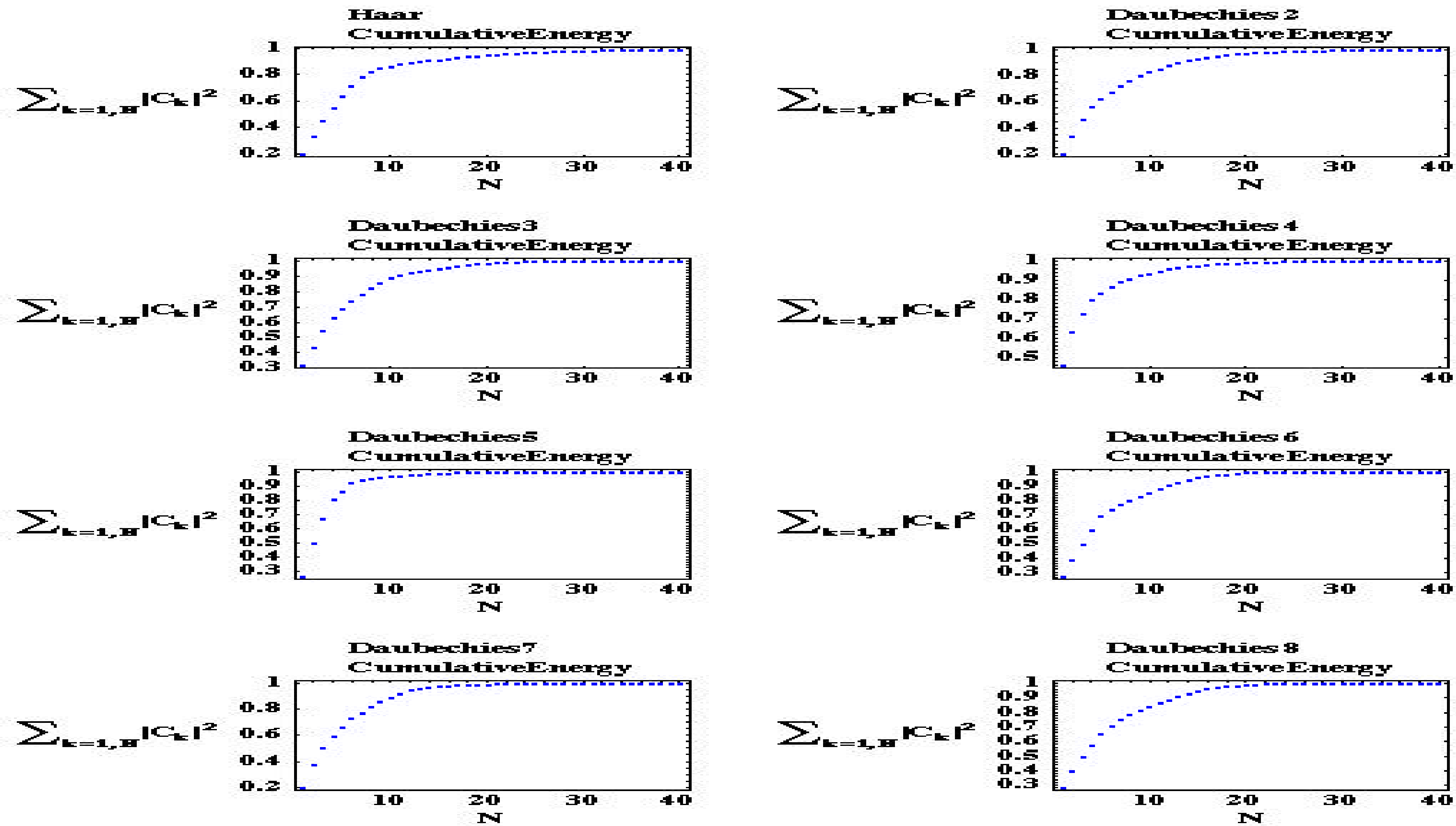




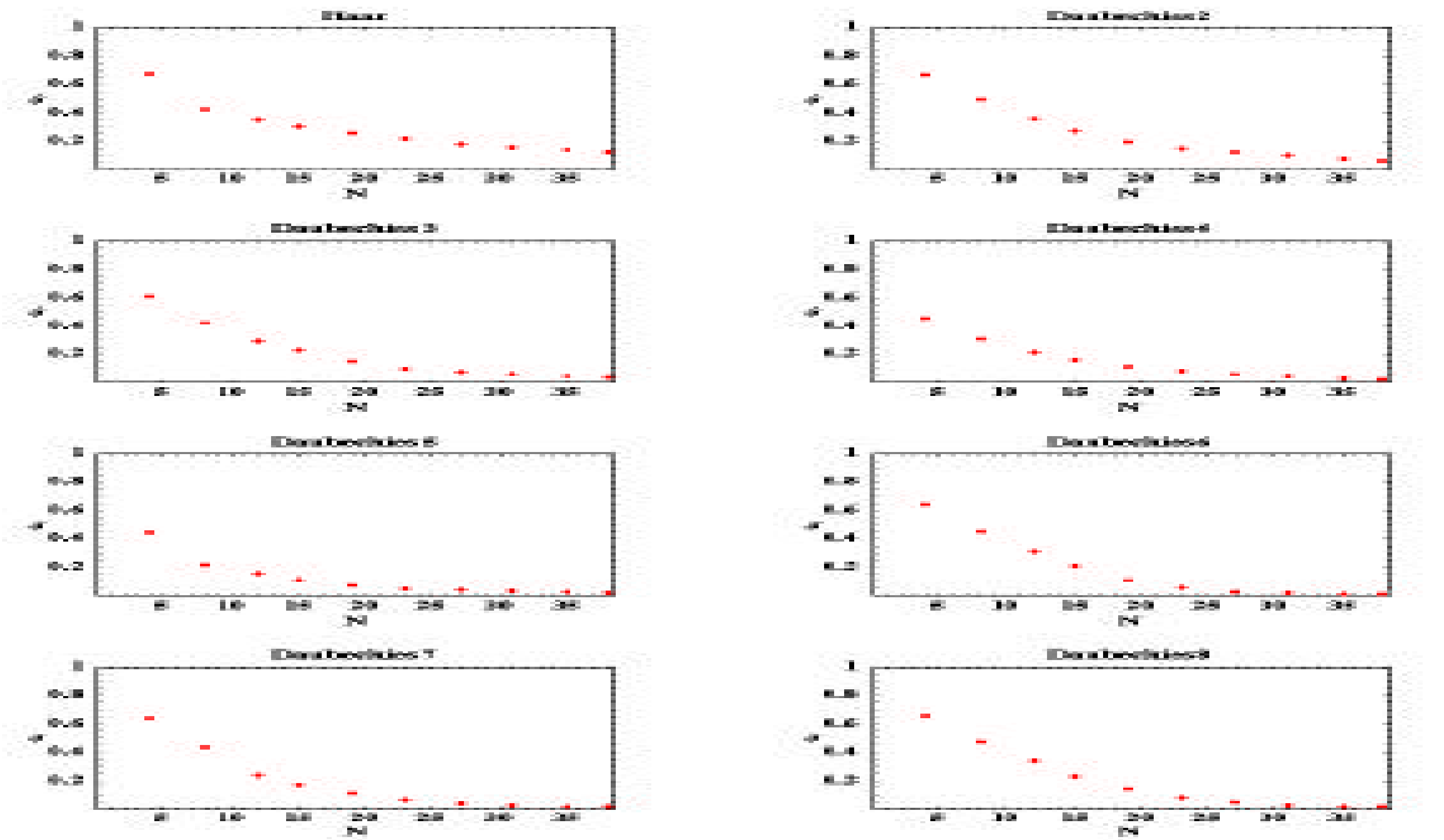




The Energy Accumulation Rate in Coefficient Space



Fractional Least Square Error vs # of Largest WLT Coefficients Kept in the Reconstruction



Least Square Error as a Function of Largest MRD Level Kept in the Reconstruction

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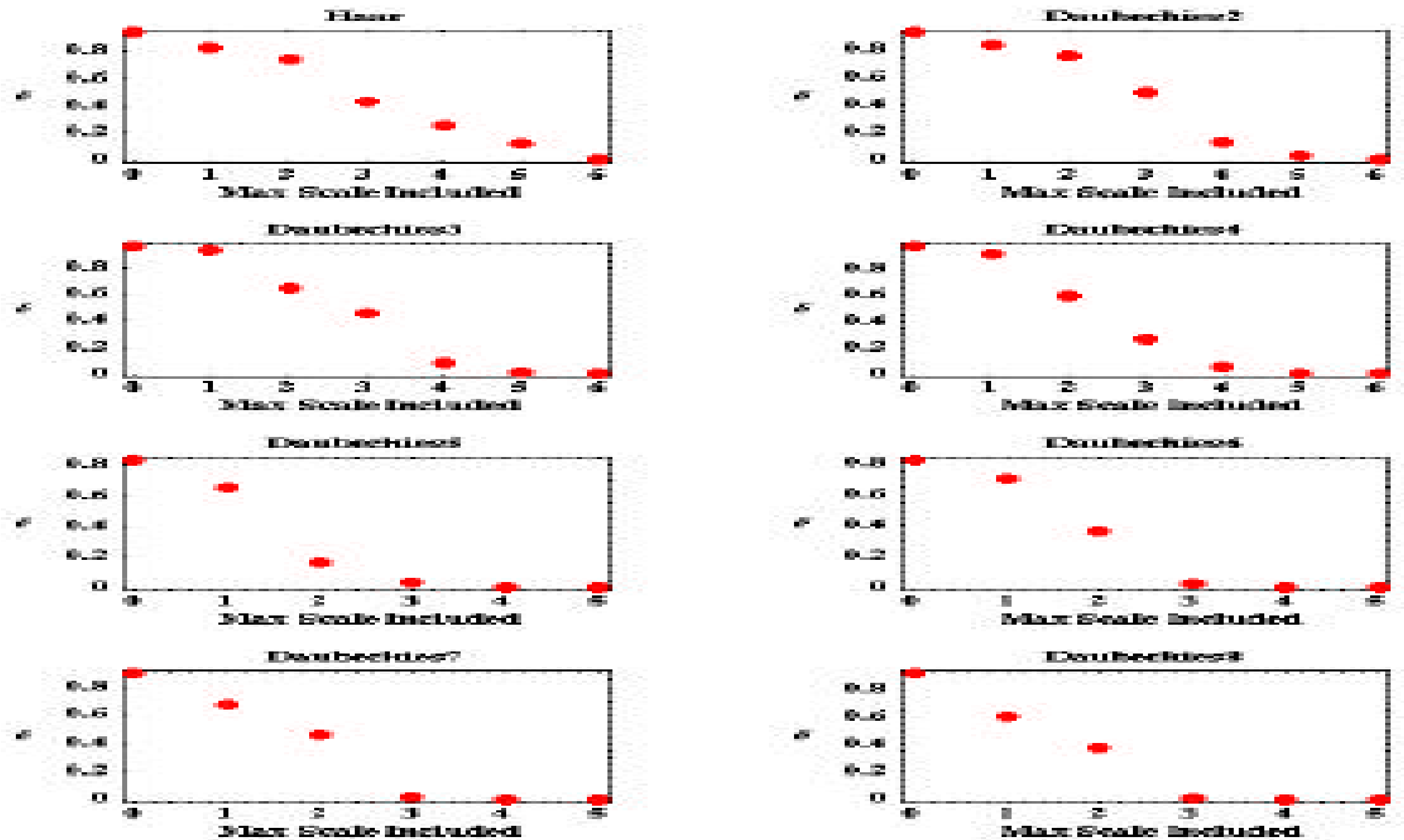
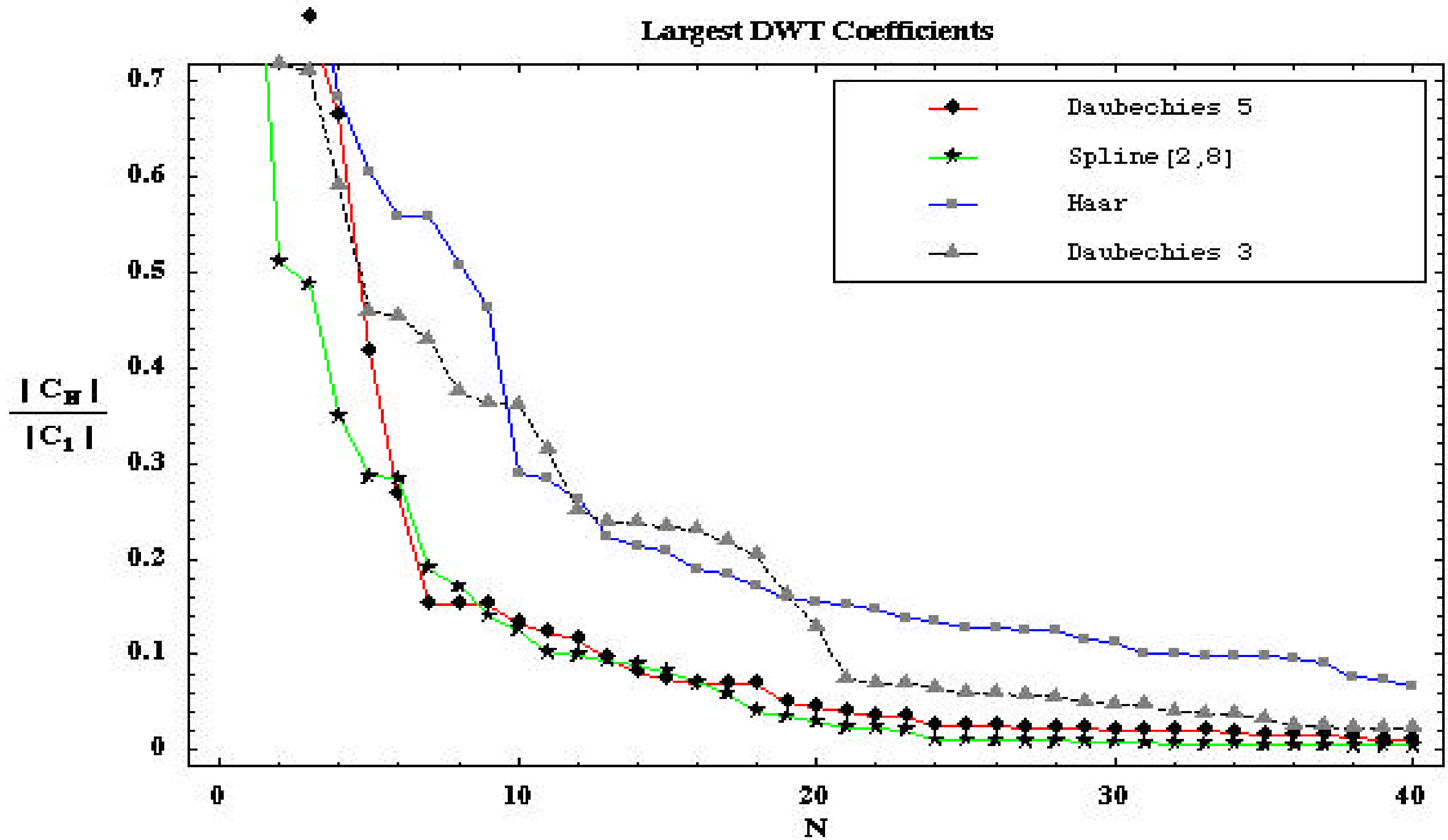
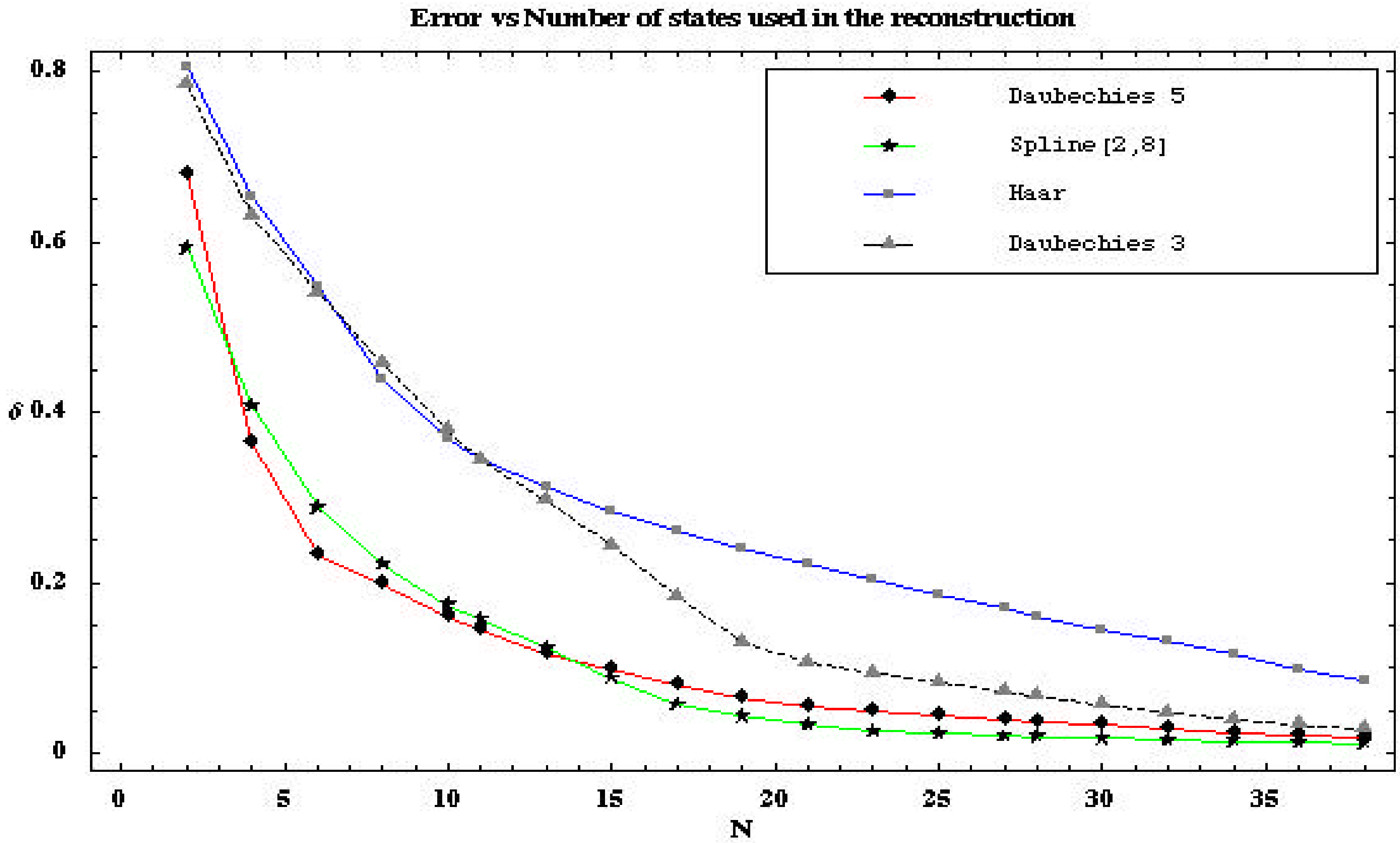


Direct Comparison of the Largest Coefficient Decay Rate vs Largest Coefficients Number

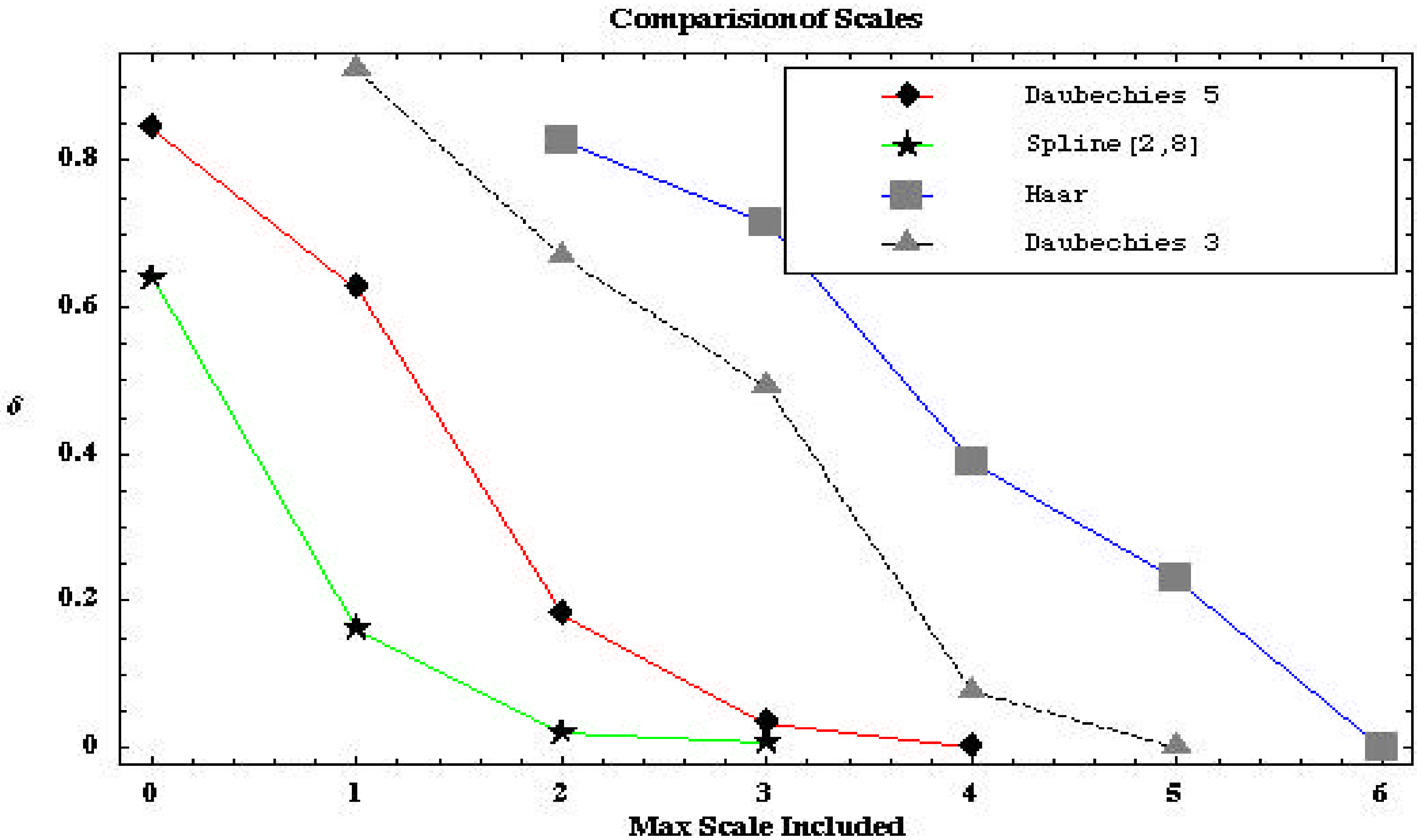


Fractional Least Square Error vs Number of Largest Wavelets Kept for Reconstraction



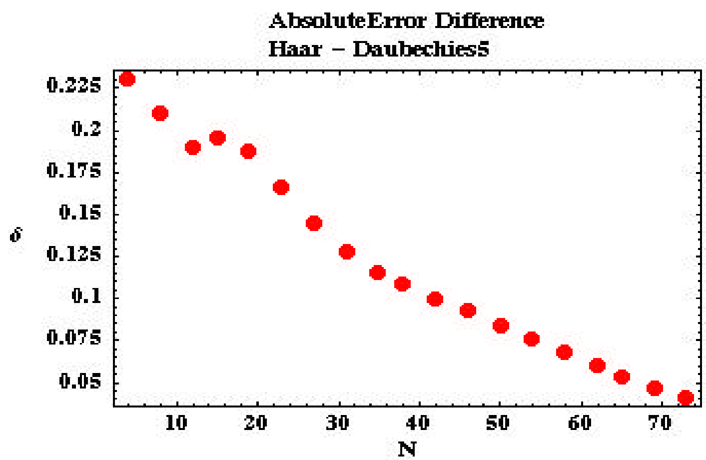
Fractional Least Square Error vs Levels of MRD Kept in the Reconstruction

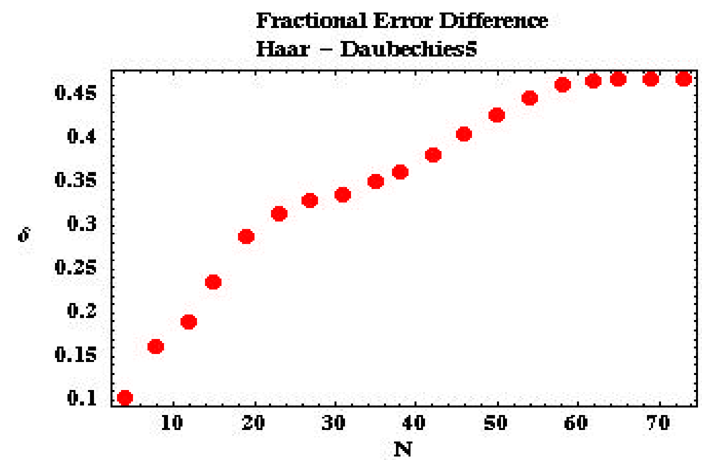
Polymath Research Inc.

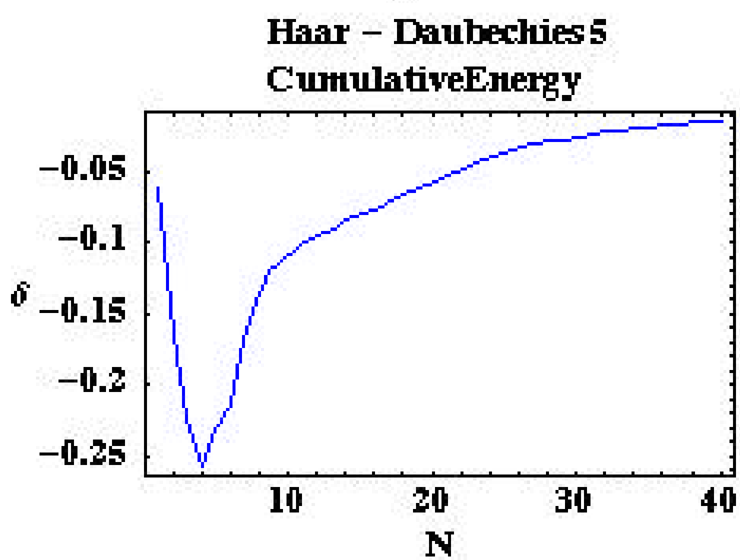


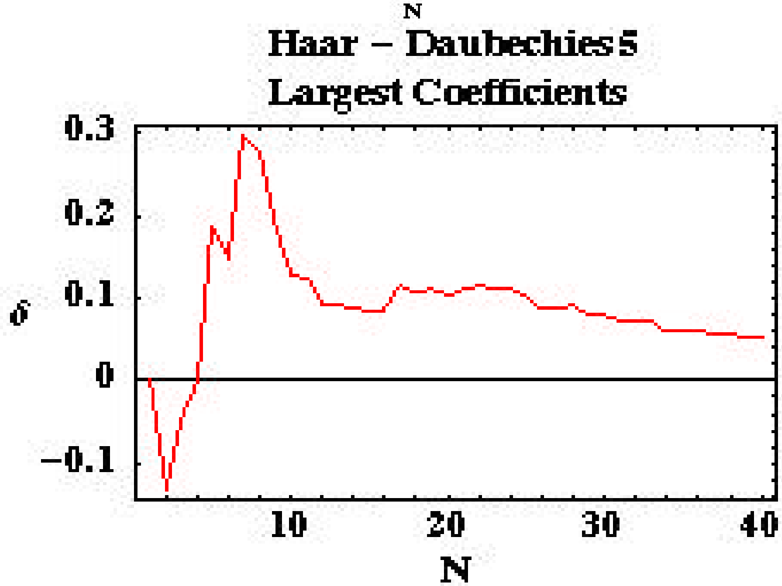
Traditional Comparisons of the Performance of 2 Wavelet Choices Haar vs Daubechies 5

Polymath Research Inc.









What Is a Suitable Phase Space Distribution Function Corresponding to a Signal f(x)? (the Wigner Function)

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The Wigner function is a bilinear functional of the signal f(x) which represents the signal in phase space-- which is to say-- simultaneously in position and wave-number space. It has almost the same form whether written in terms of f(x) or f_{wiggle} (k):

$$W_f(x,k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx'\, e^{-ikx'}\, f\left(x + \frac{x'}{2}\right) f^*\left(x - \frac{x'}{2}\right)$$

$$W_f(x,k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk'\, e^{-ik'x}\, \hat{f}^*\left(k + \frac{k'}{2}\right) \hat{f}\left(k - \frac{k'}{2}\right)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dk\, W_f(x,k) = \int_{-\infty}^{\infty} \left|f(x)\right|^2 dx = \int_{-\infty}^{\infty} \left|\hat{f}(k)\right|^2 dk$$

WRMR Analysis: MRD Seen from the POV of Wigner Transforms

- Discrete Wavelet Transforms allow the multi-resolution decomposition of a signal in wavelet bases which while constituting a complete or over-complete basis can be used to generate a series of nested approximations of that signal.
- One such sequence of approximations involves thresholding the wavelet series by keeping an ordered set of largest coefficients.
- Another sequence of approximations is generated by keeping successively finer scale or detail levels of the MRD.
- In both cases, the performance of different wavelet families can be compared to each other by using the Wigner phase space representations of these nested sets of approximations (WRMR analysis) to see which of them reproduces desired features of the Wigner representation of the signal most closely with a few terms kept or most quickly as more terms are added.

Why Should WRMR Analysis Work or When?

- The Wigner transform is a nonlinear (quadratic) functional of the amplitude of the signal. But its functional form is the same in x or k space. It is a democratic quadratic functional which is a simultaneous convolution (correlation function kernel)and Fourier transform.
- Expect weak or bad representations of the signal via some thresholding of a DWT MRD to appear far off in its Wigner transform and thus lead to better discrimination from wavelet families that do capture the essence of the Wigner representation of the signal with a few coefficients or levels of MRD.
- The idea is to make the acceptance criteria of some wavelet family vs another be based on pattern recognition in Wigner transformed phase space.
- Besides data compression and pattern recognition,denoising via WRMR analysis has interesting possibilities in Wigner phase space.

Consider a Signal which is a "Sum of N Gaussian Complex Wavepackets"

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$$\varphi(x) = \sum_{i=1}^{N} |\varphi_i| e^{i\theta_i} e^{ik_i x} e^{-(x-x_{ic})^2 / x_{iw}^2}$$

The Wigner function representation of ϕ(x) in phase space is given by:

$$W_\varphi(x,k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \, e^{-ikx'} \varphi^*(x - x'/2) \varphi(x + x'/2)$$

The Wigner Function Representation of the "Sum of N Gaussian Complex Wavepackets" Is Given By

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$$W_\varphi(x,k) = \sum_{i=1}^{N} \frac{x_{iw}}{\sqrt{2\pi}} |\varphi_i|^2 \exp\left[-2(x-x_{ic})^2/x_{iw}^2\right] \exp\left[-(k-k_i)^2 x_{iw}^2/2\right]$$

$$+ \sum_{i=1}^{N}\sum_{j>i}^{N} \frac{1}{\sqrt{\pi\left(\frac{1}{x_{iw}^2}+\frac{1}{x_{jw}^2}\right)}} |\varphi_i|\,|\varphi_j| \times$$

$$\times \exp\left[-\frac{\left[(x-x_{ic})+(x-x_{jc})\right]^2}{x_{iw}^2+x_{jw}^2}\right] \times$$

$$\times \exp\left[-\left[k-\frac{(k_i+k_j)}{2}\right]^2 \Big/ \left(\frac{1}{x_{iw}^2}+\frac{1}{x_{jw}^2}\right)\right] \times$$

$$\times 2\cos\left[\frac{\left(2k-(k_i+k_j)\right)\left(\frac{(x-x_{ic})}{x_{iw}^2}-\frac{(x-x_{jc})}{x_{jw}^2}\right)}{\left(\frac{1}{x_{iw}^2}+\frac{1}{x_{jiw}^2}\right)} + \left(\theta_i-\theta_j+(k_i-k_j)x\right)\right]$$

The Wigner Function Representation of the Gaussian Enveloped Two Separate Carriers Signal

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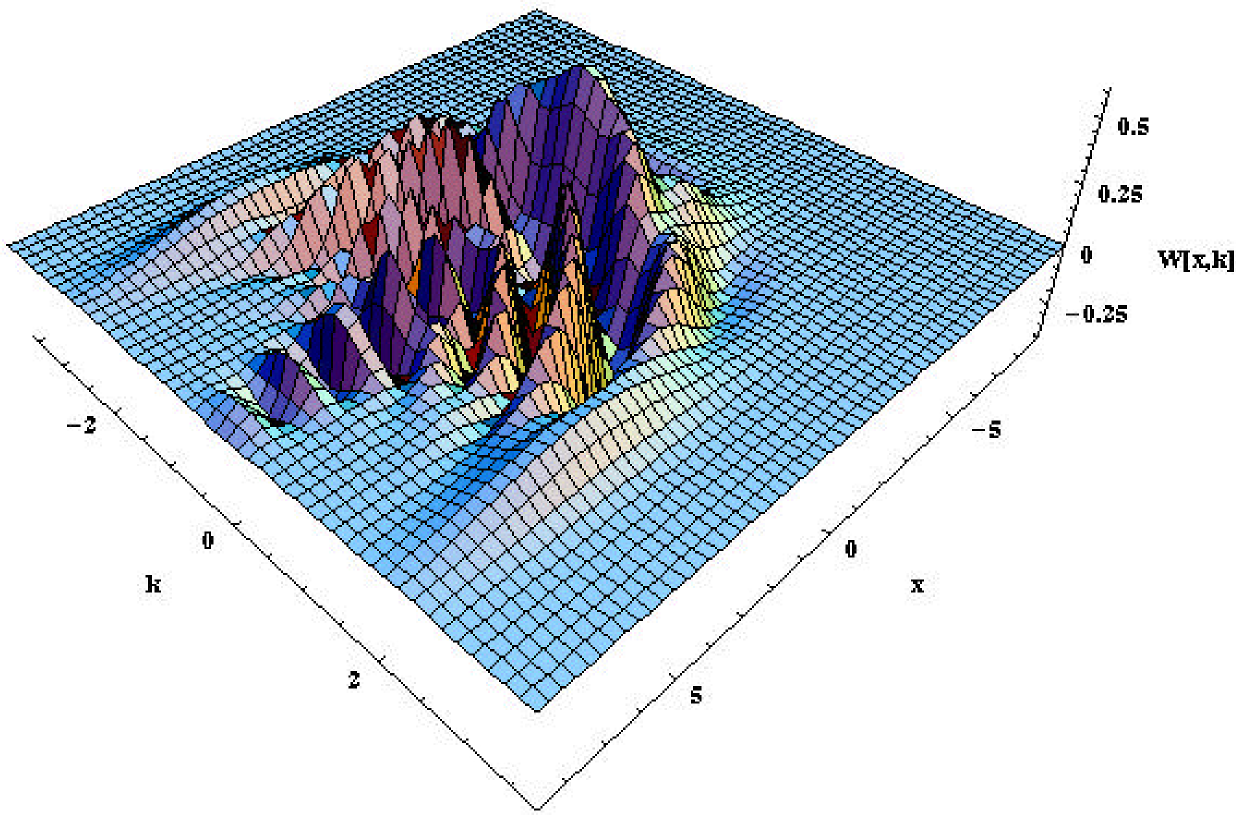
Wigner Transform of Original Data



The Wigner Function of One of the Gaussians with a Low Frequency Carrier

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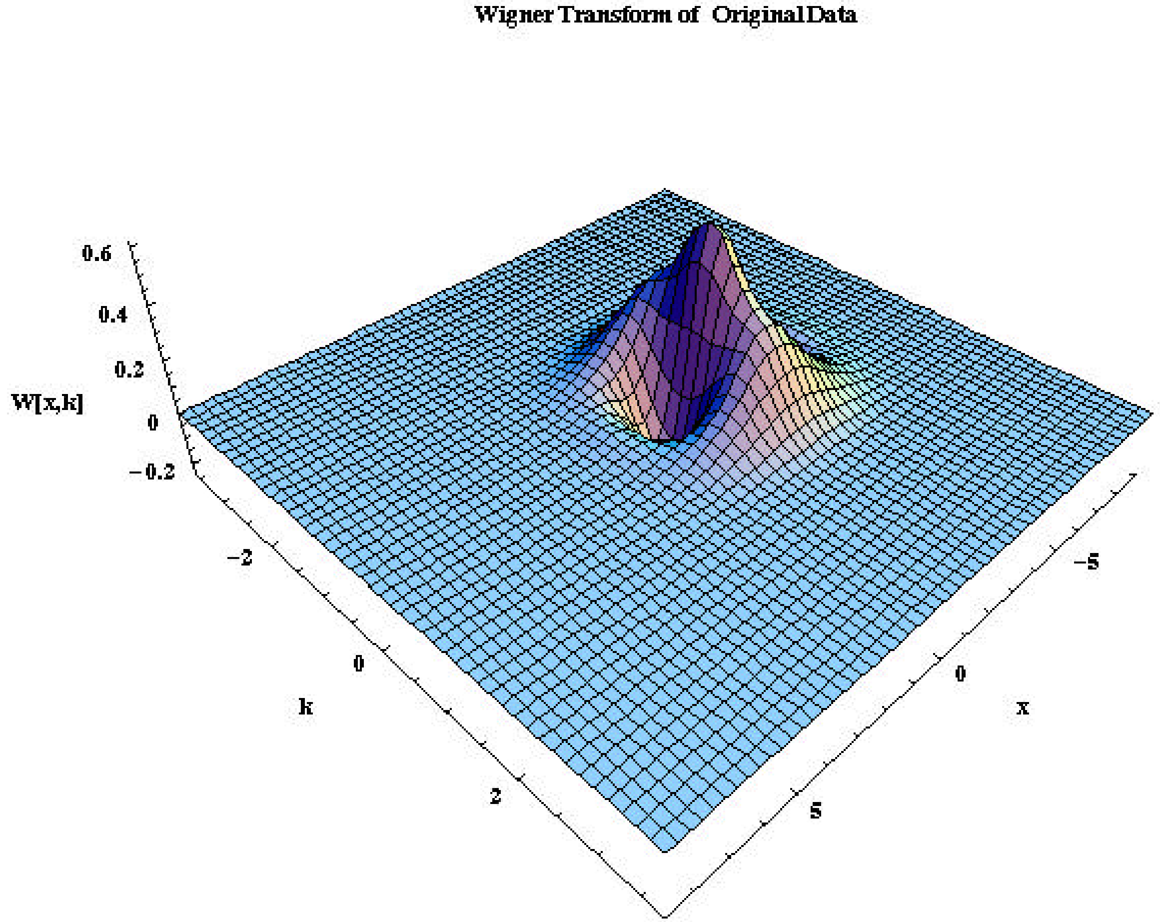
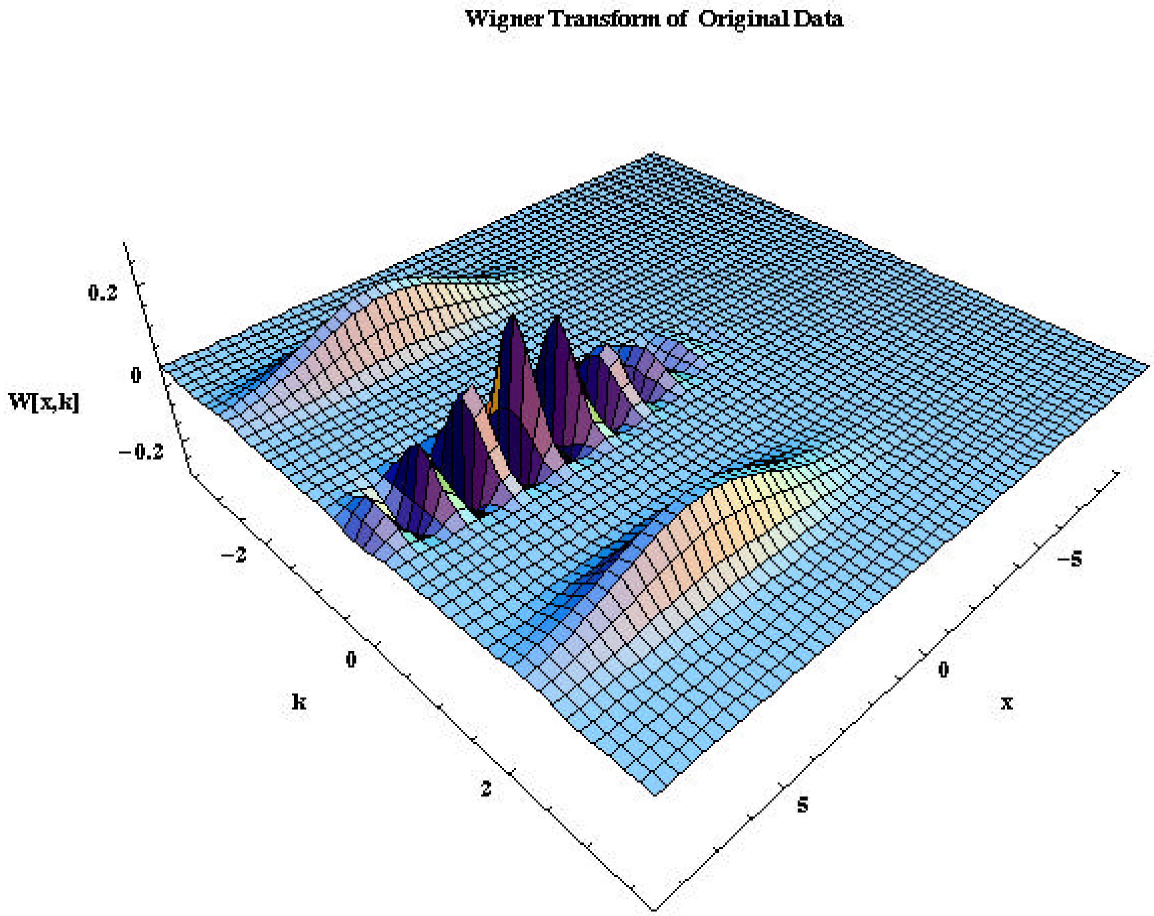
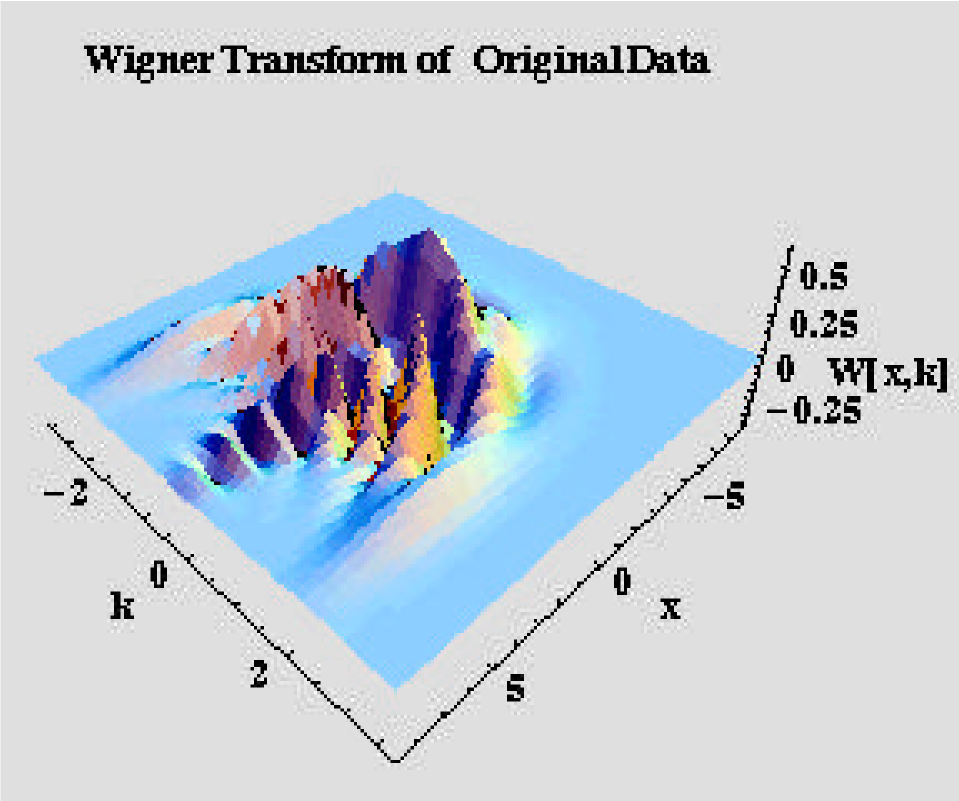


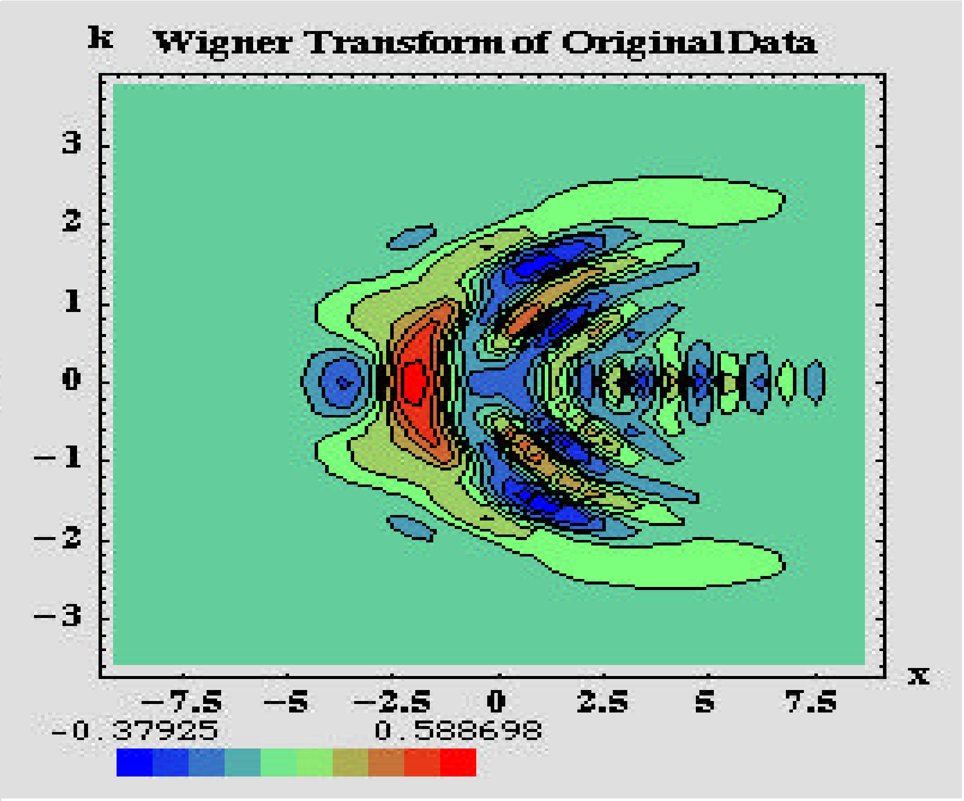
The Wigner Function of the Other Gaussian with a High frequency Carrier

Polymath Research Inc.



Wigner Transform of the Double Gaussian Wavepacket





Mean Square Error in Haar MRD's Performance in Wigner Land as We Add Largest Coeffs (5/75%, 10/46%,15/35%, 20/29%, 25/24%)

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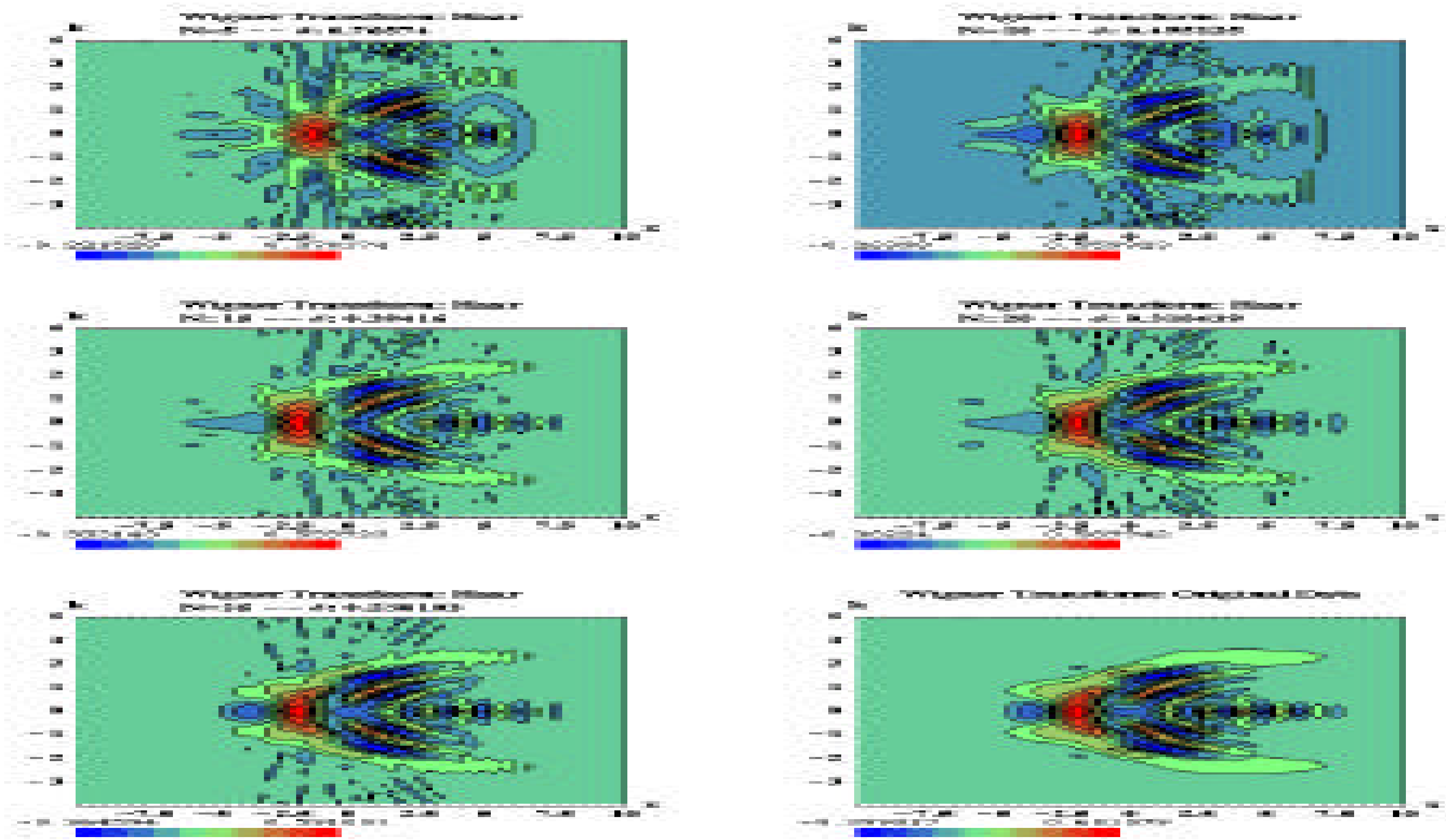

Mean Square Error in Haar MRD's Performance in Wigner Land as We Add Finer Levels (L1/99.8%, L2/99.2%, L3/94.5%, L4/86.1%)

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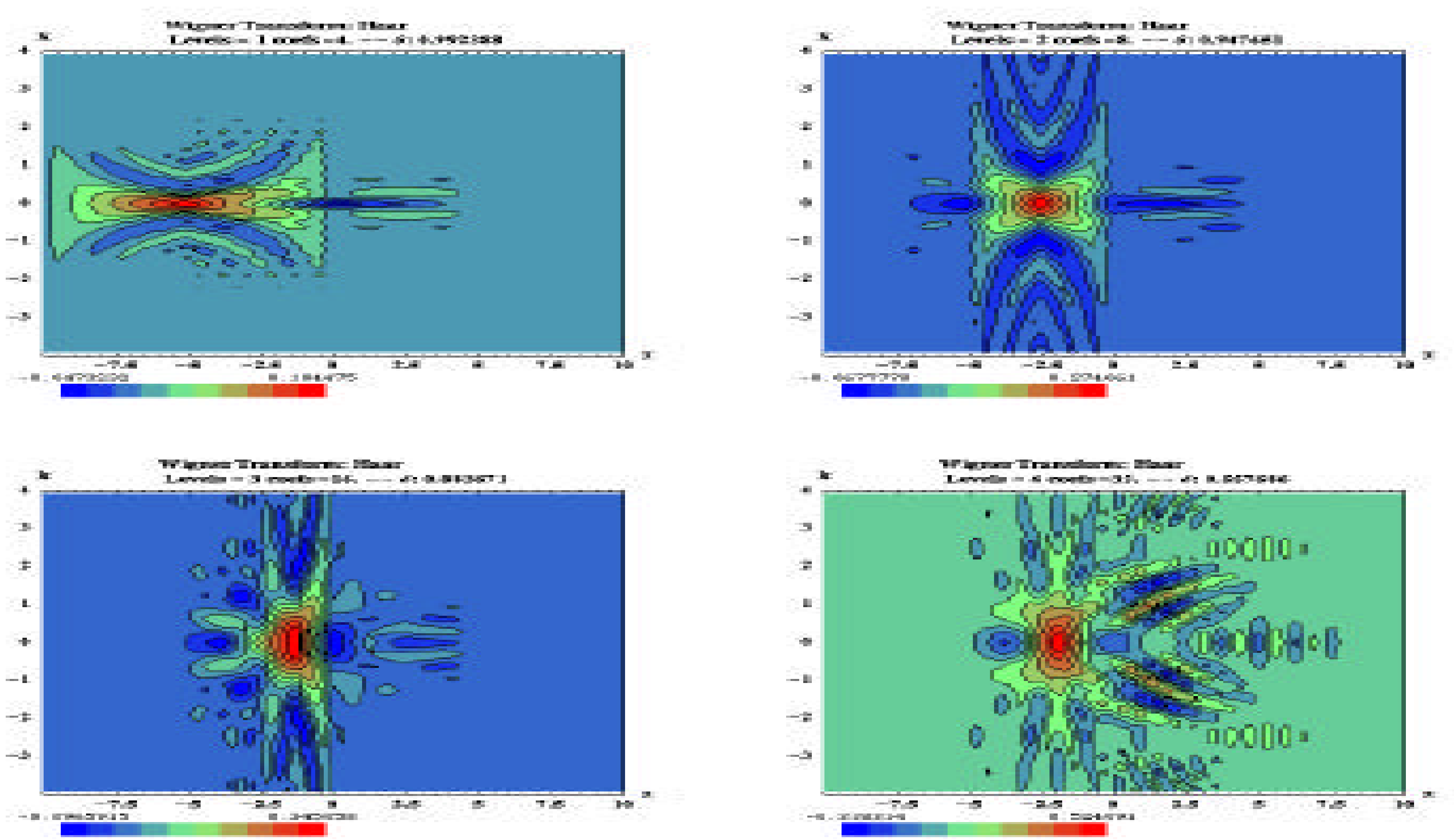
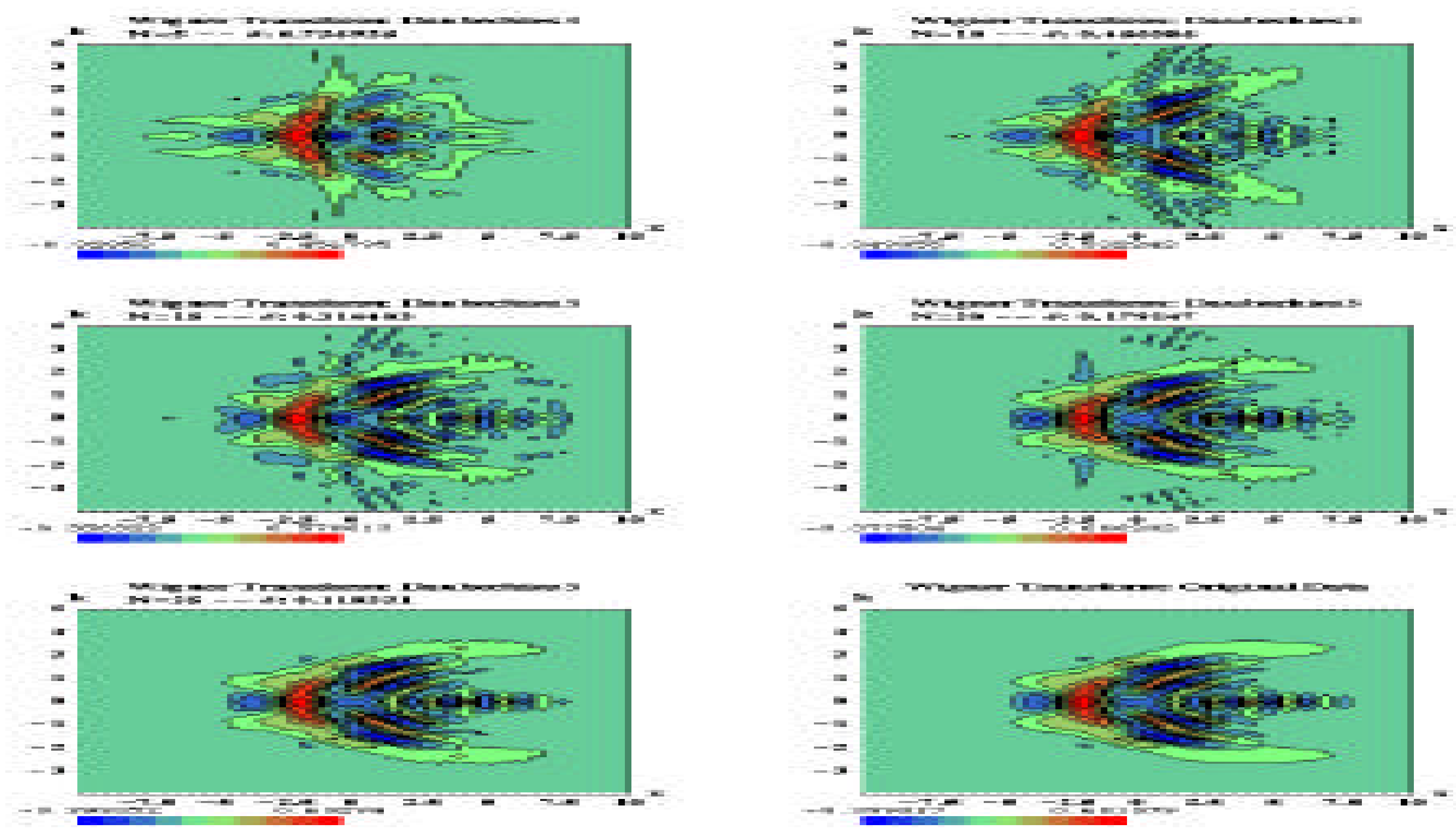


Mean Square Error in Daub3 MRD's Performance in Wigner Land as We Add Largest Coeffs (5/73%, 10/47%, 15/31%, 20/18%, 25/12%)



Mean Square Error in Daub3 MRD's Performance in Wigner Land as We Add Finer Levels (L1/99.6%, L2/99%, L3/81%, L4/61%

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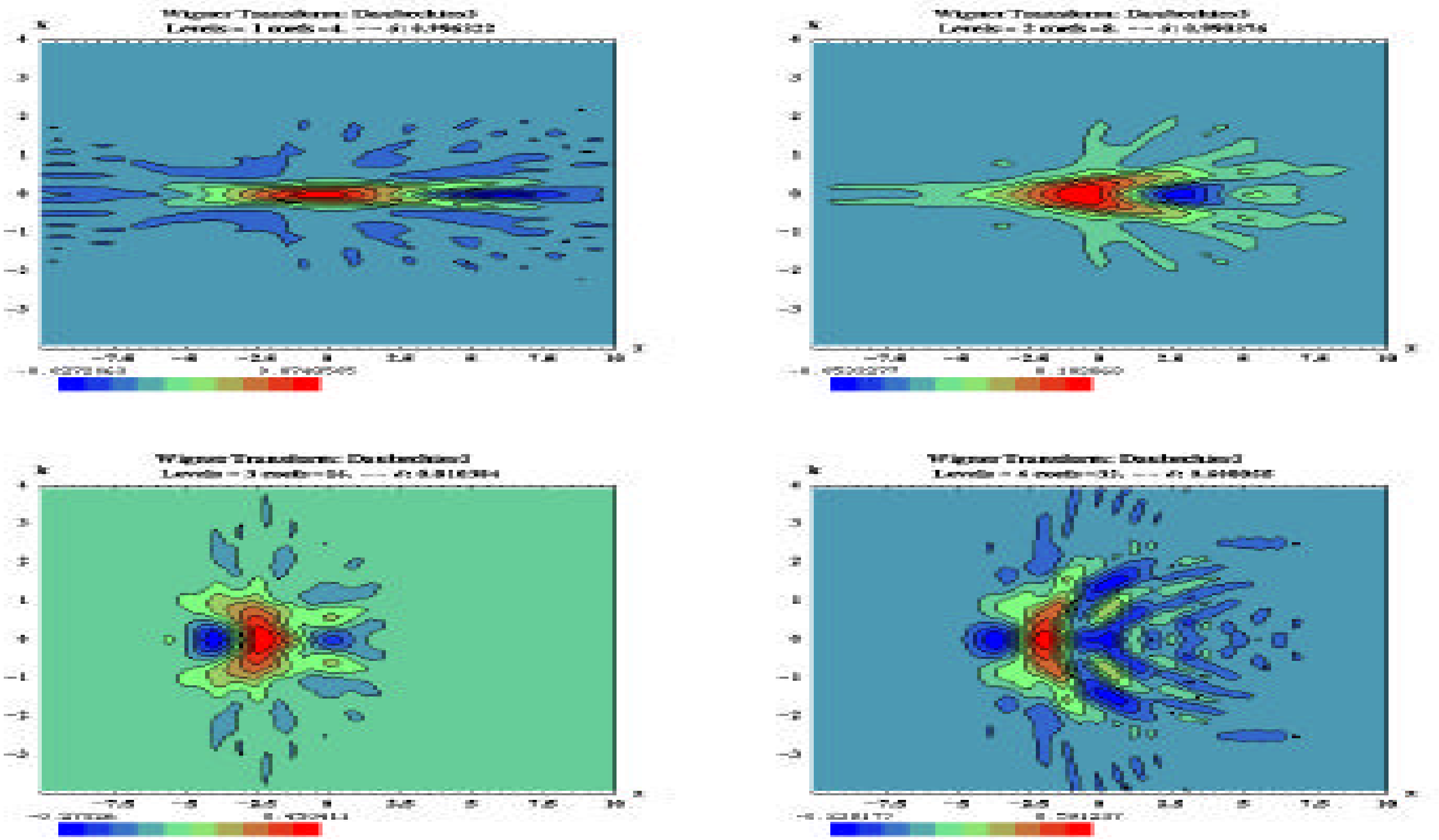

Largest Coefficient WRMR Analysis with Daubechies 5 Wavelets (5/51%,10/26%,15/15%, 20/9%, 25/5.8%)

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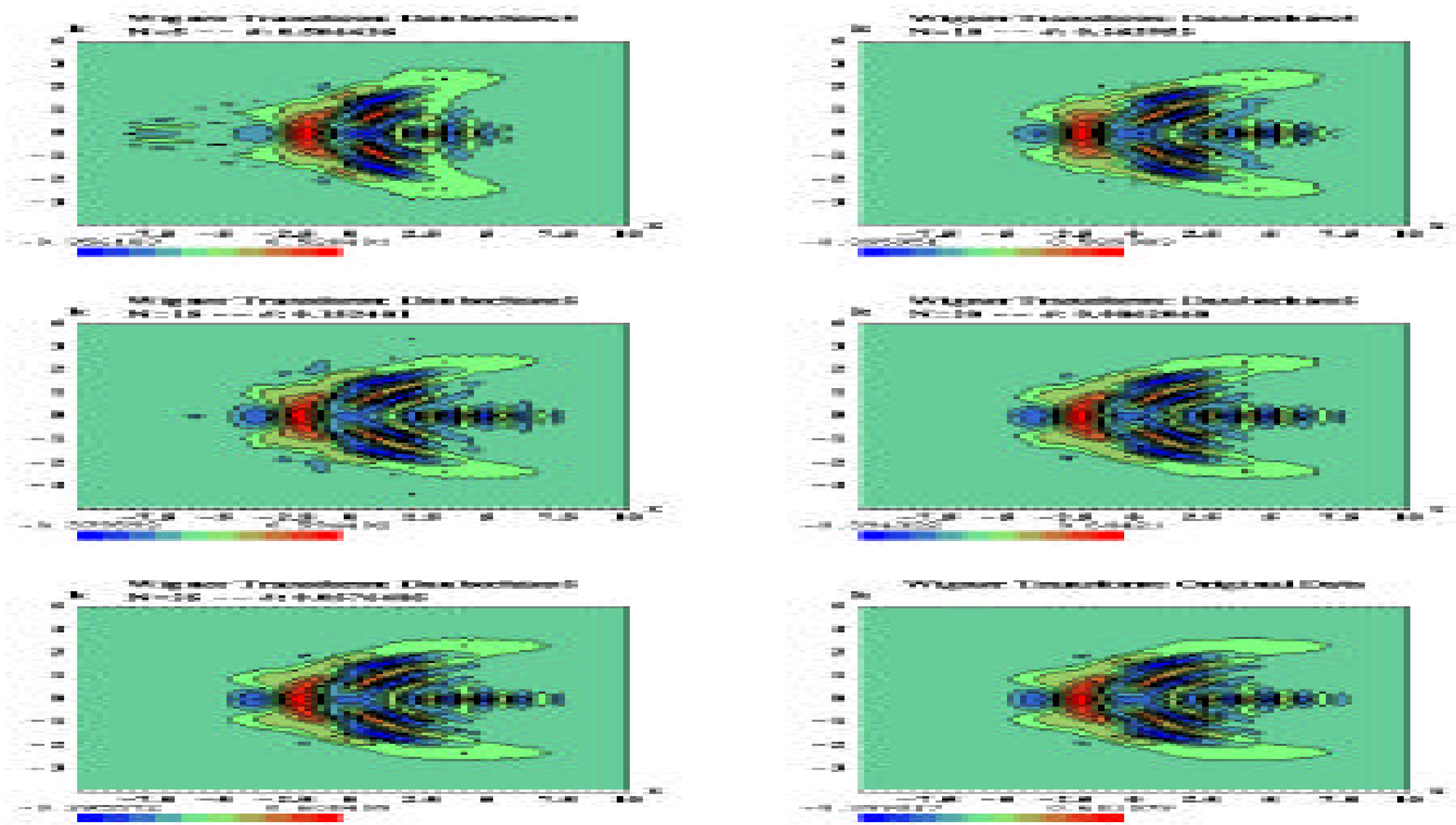
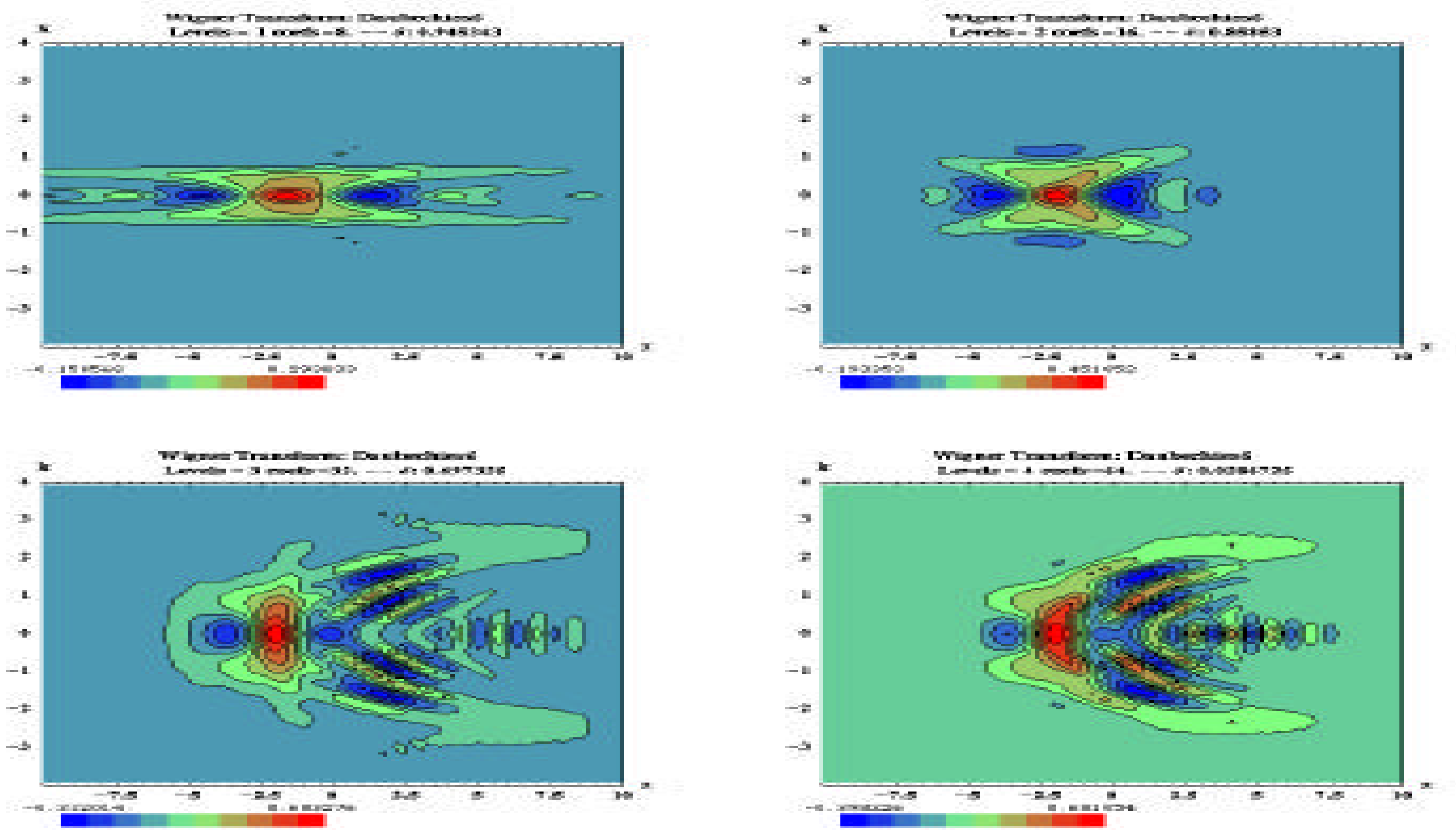


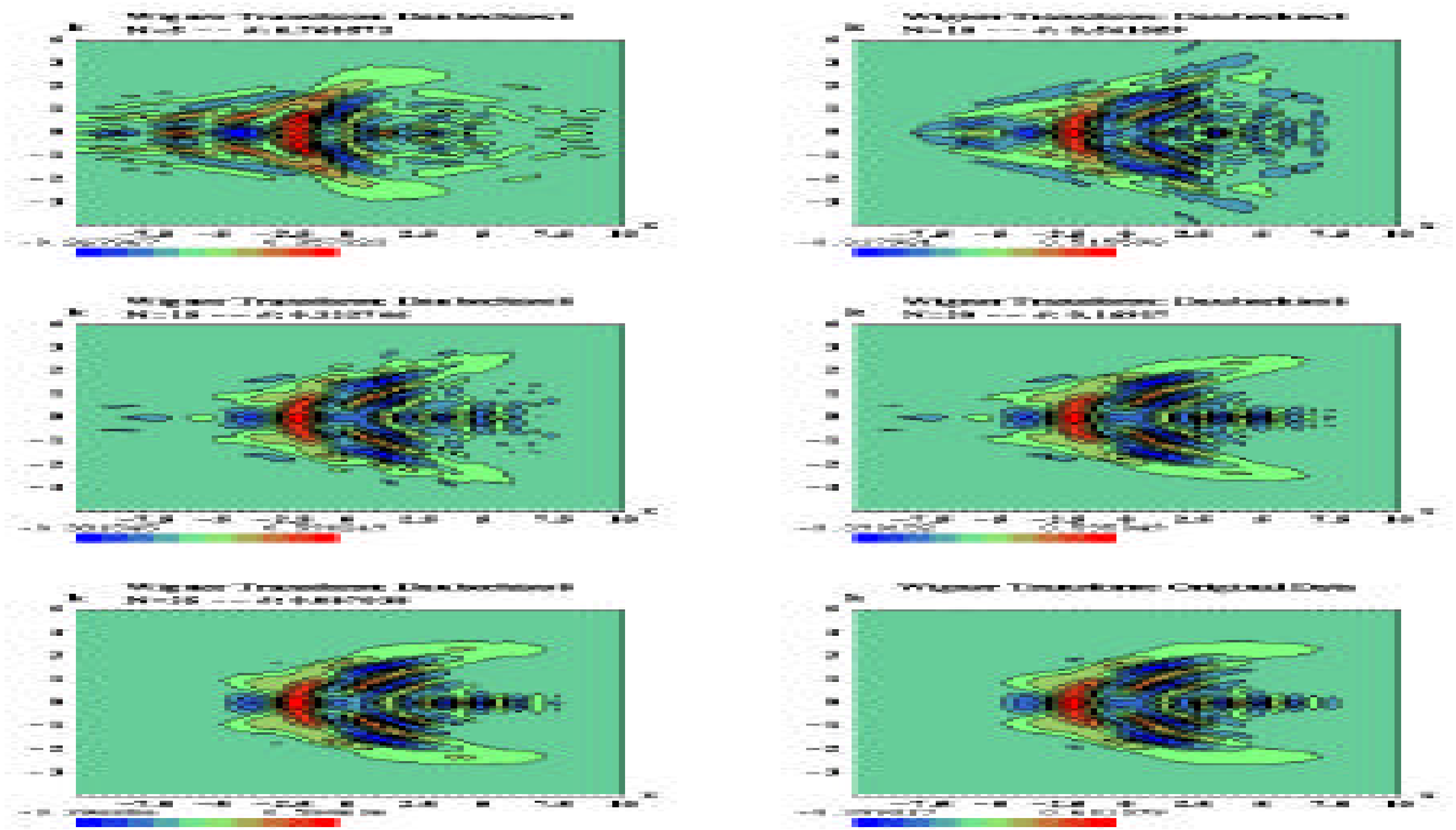
Level by MRD Level WRMR Analysis Using Daub6

(L1/94%, L2/85%, L3/50%, L4/2.8%)



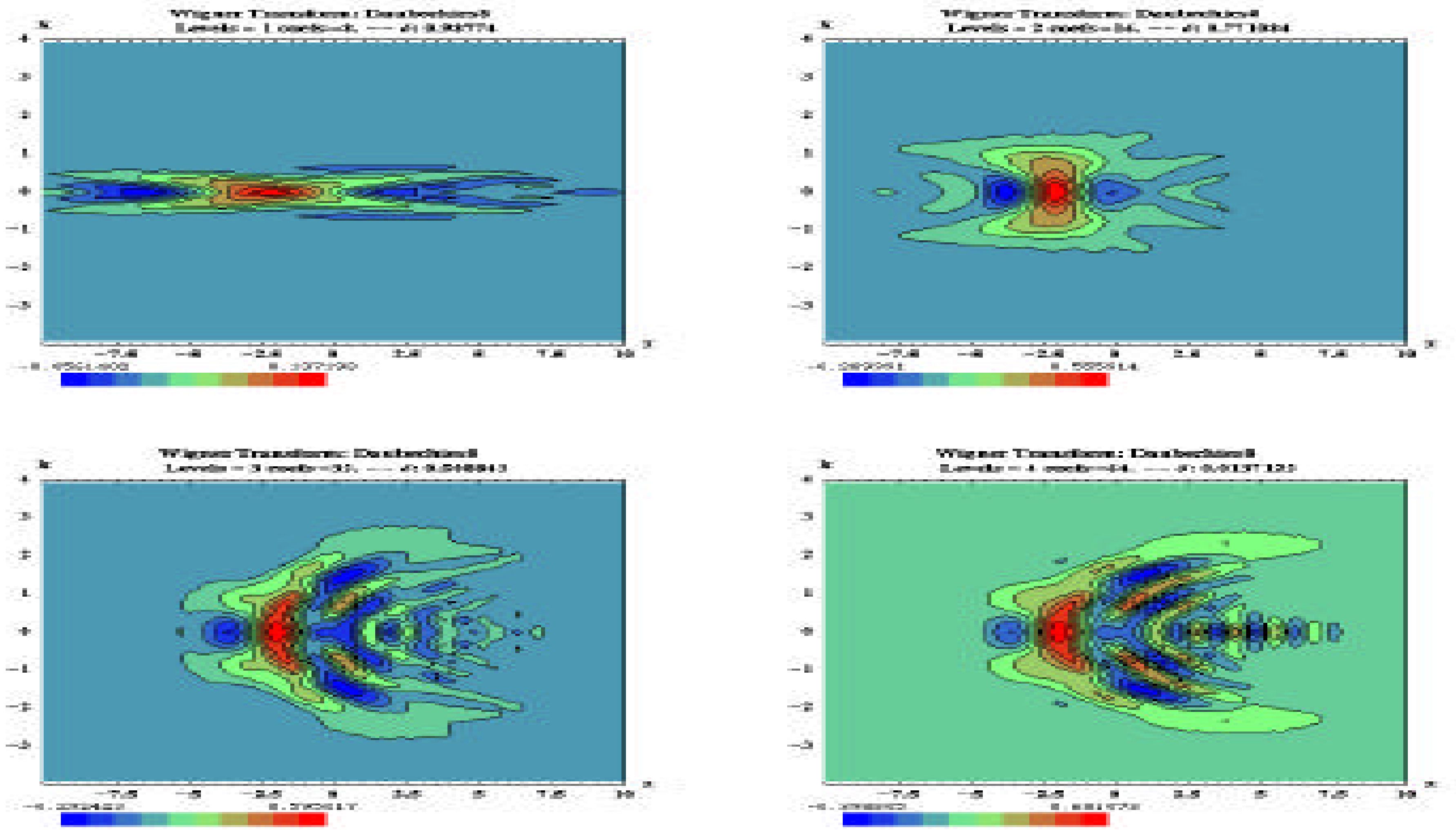
Largest Coefficient WRMR Analysis with Daub8 Wavelets

(5/76%, 10/54%, 15/33%, 20/17%, 25/9%)



Level by MRD Level WRMR Analysis Using Daub8

(L1/99%,L2/77%, L3/51%, L4/14%)



Wigner Function Representation of the MRD Using LADF4

(5/79%, 10/54%, 15/34%, 20/16%, 25/9%)

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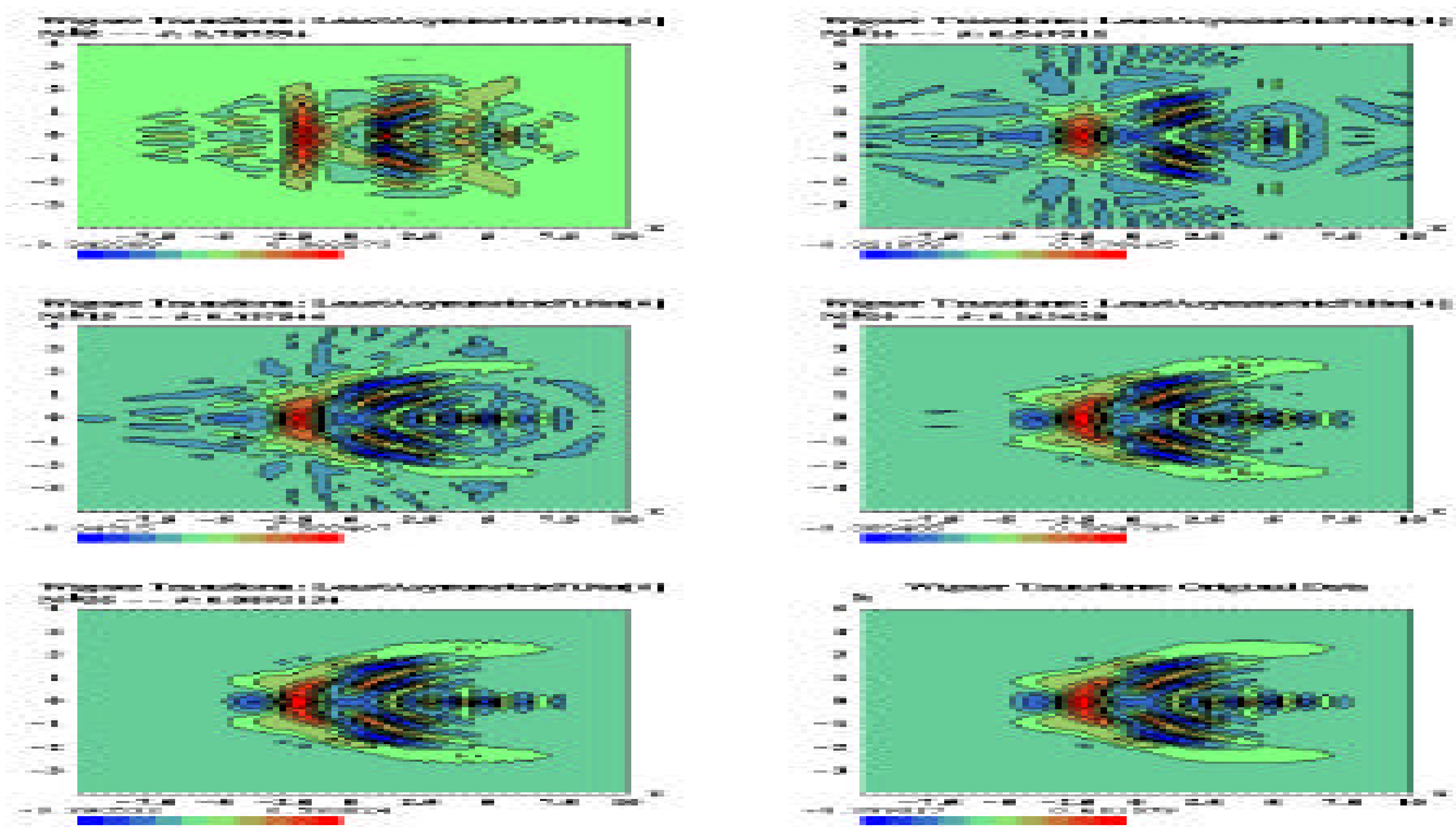
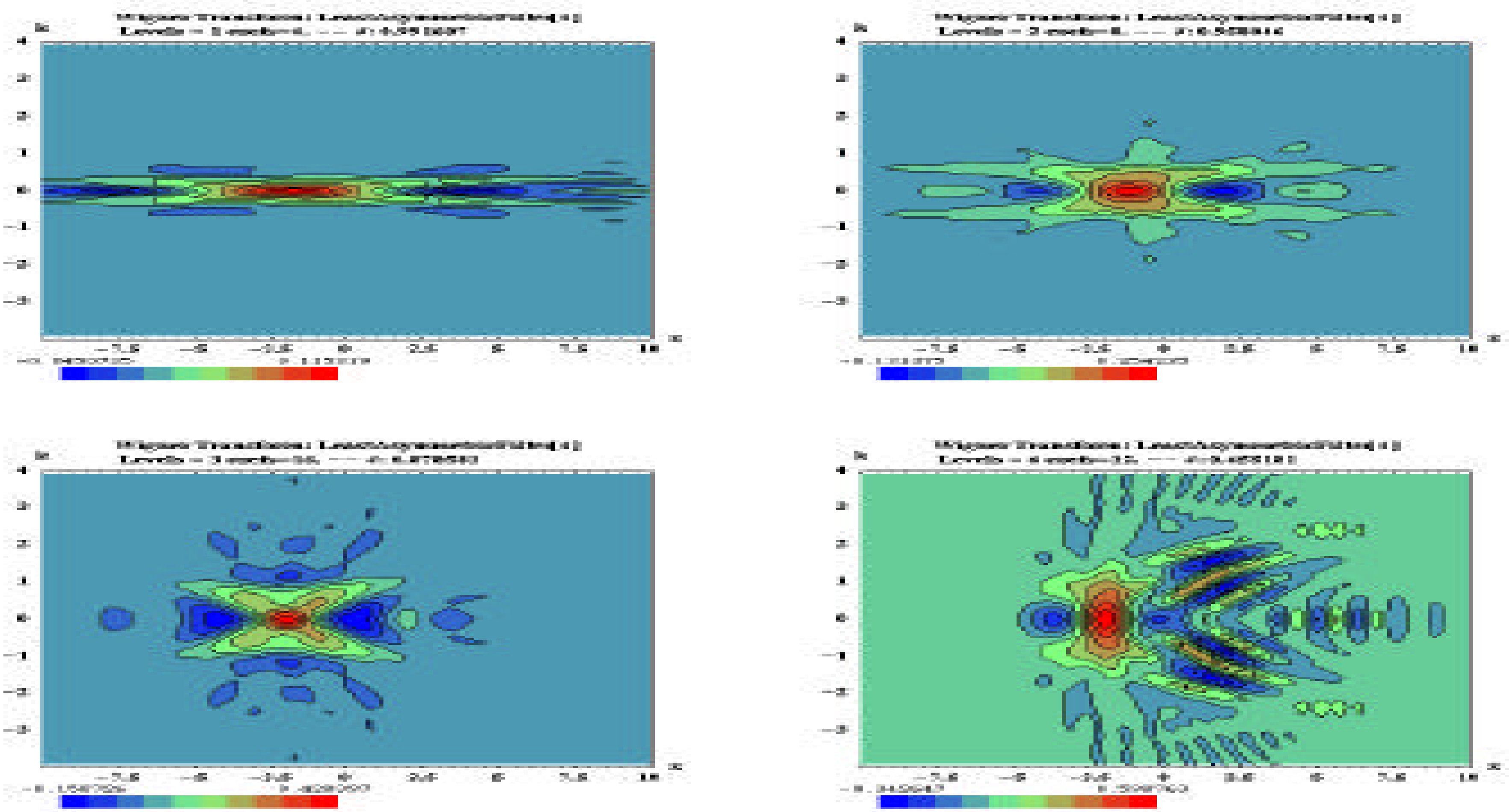


Level by MRD Level WRMR Analysis Using LADF4

(L1/99%, L2/96%, L3/88%, L4/46%)

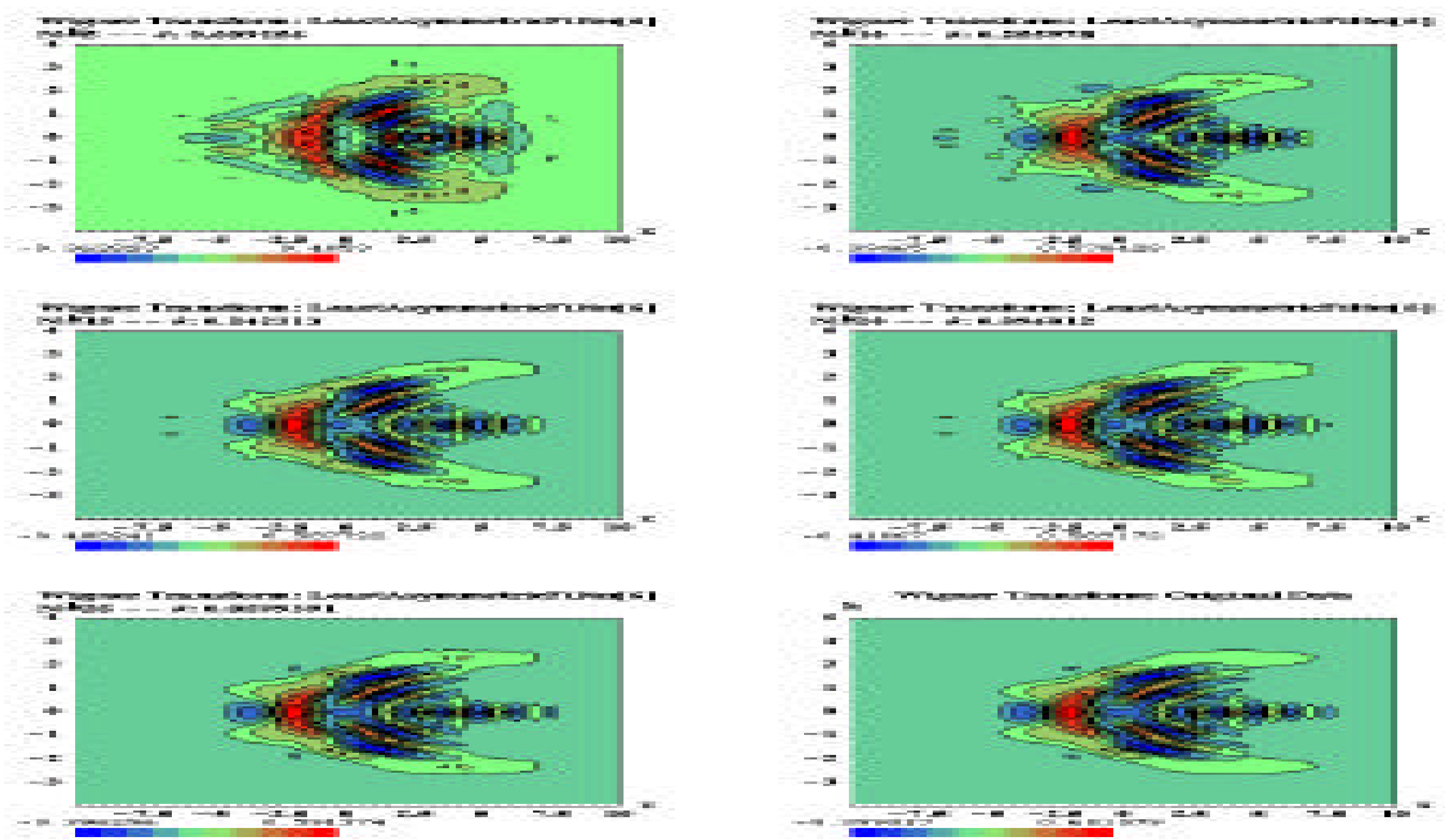
Polymath Research Inc.



Wigner Function Representation of the MRD Using LADF6

(5/65%, 10/26%, 15/14%, 20/10%, 25/6%)

Polymath Research Inc.



Level by MRD Level WRMR Analysis Using Quadratic Spline [2,4]

(L1/95%, L2/81%, L3/34%, L4/7.2%)

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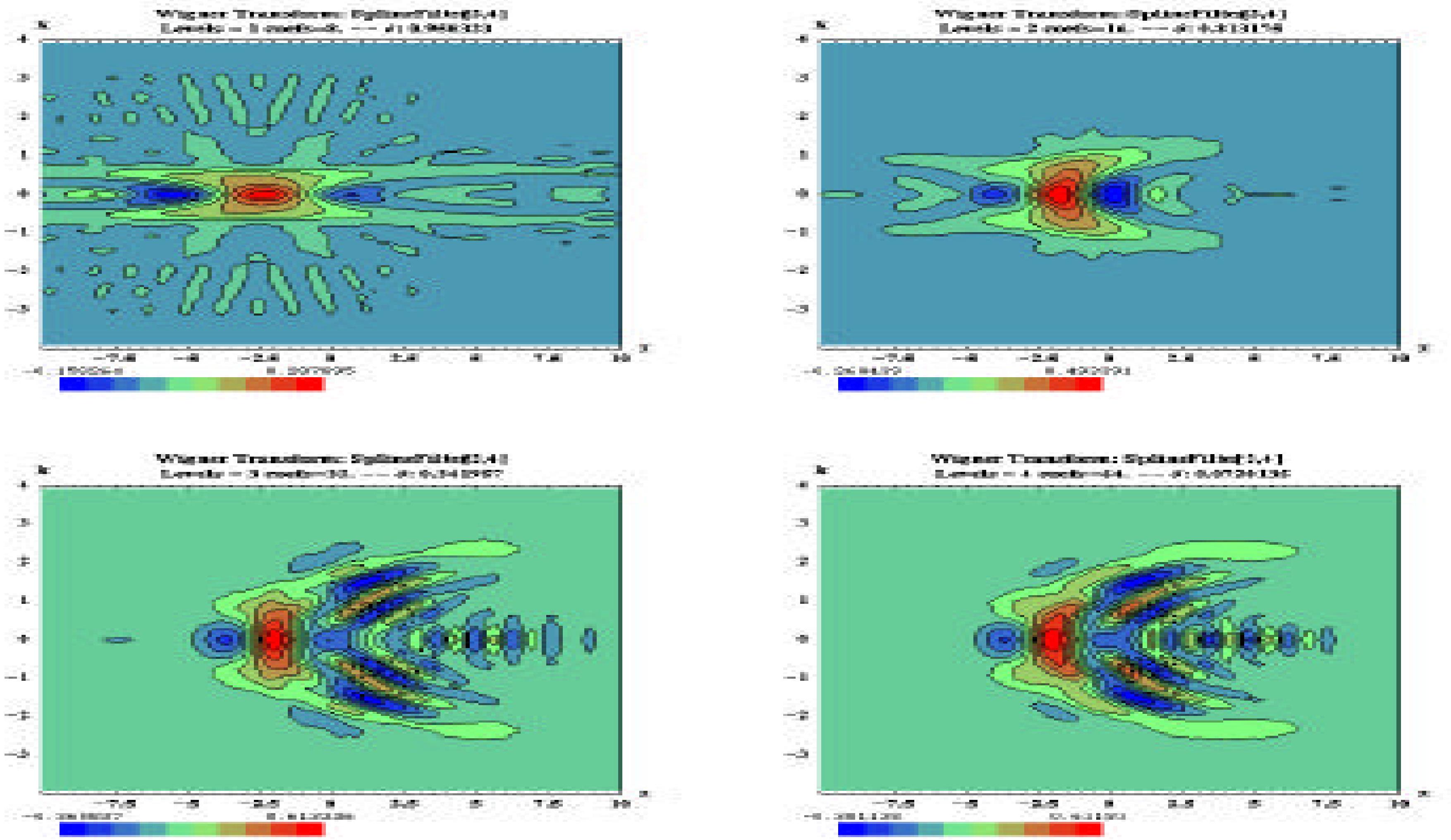

Mean Square Error in Spline[4,4] MRD's Performance in Wigner Land as We Add Largest Coeffs (5/72%, 10/40%,15/29%, 20/24%, 25/19%)

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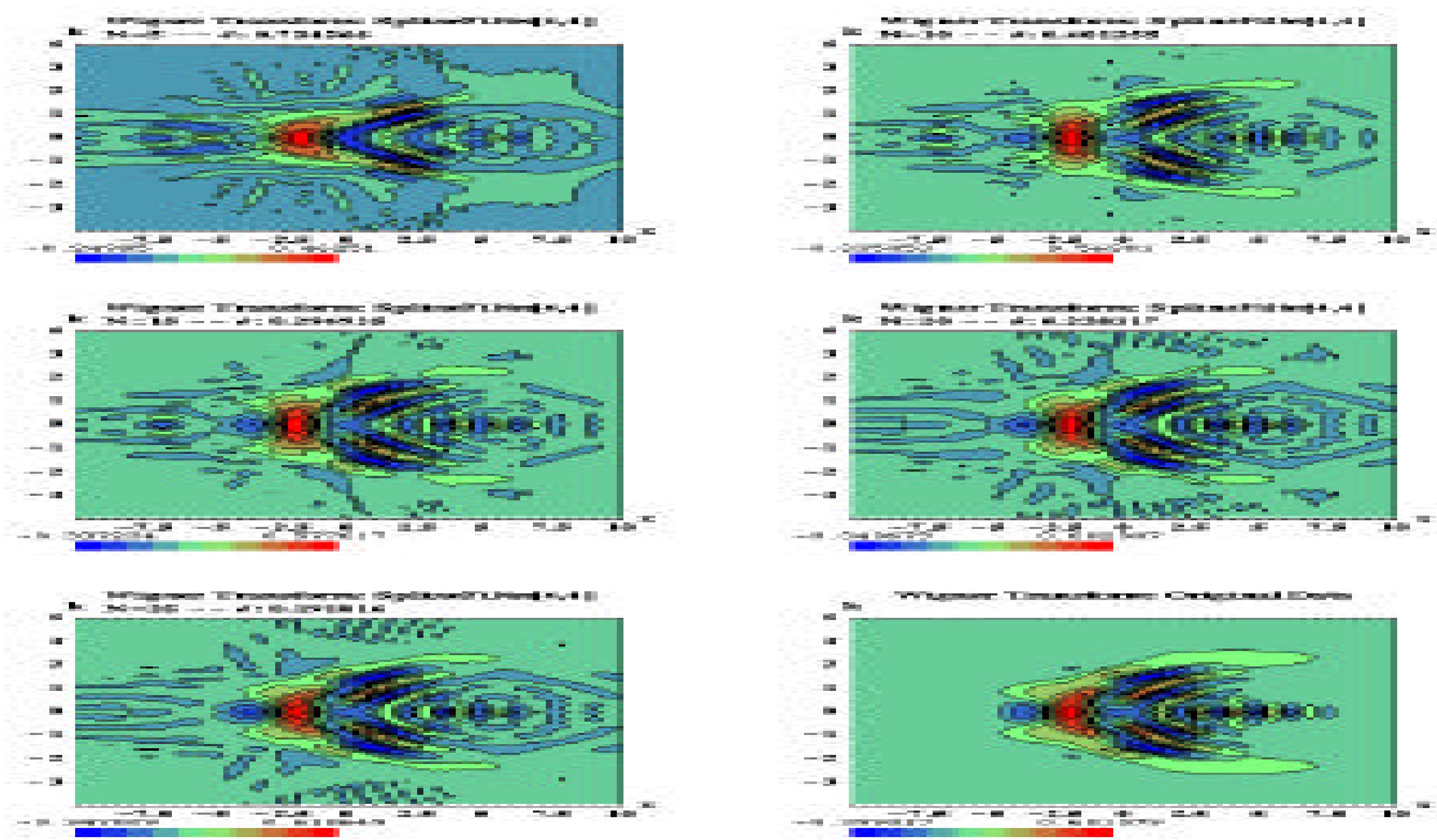
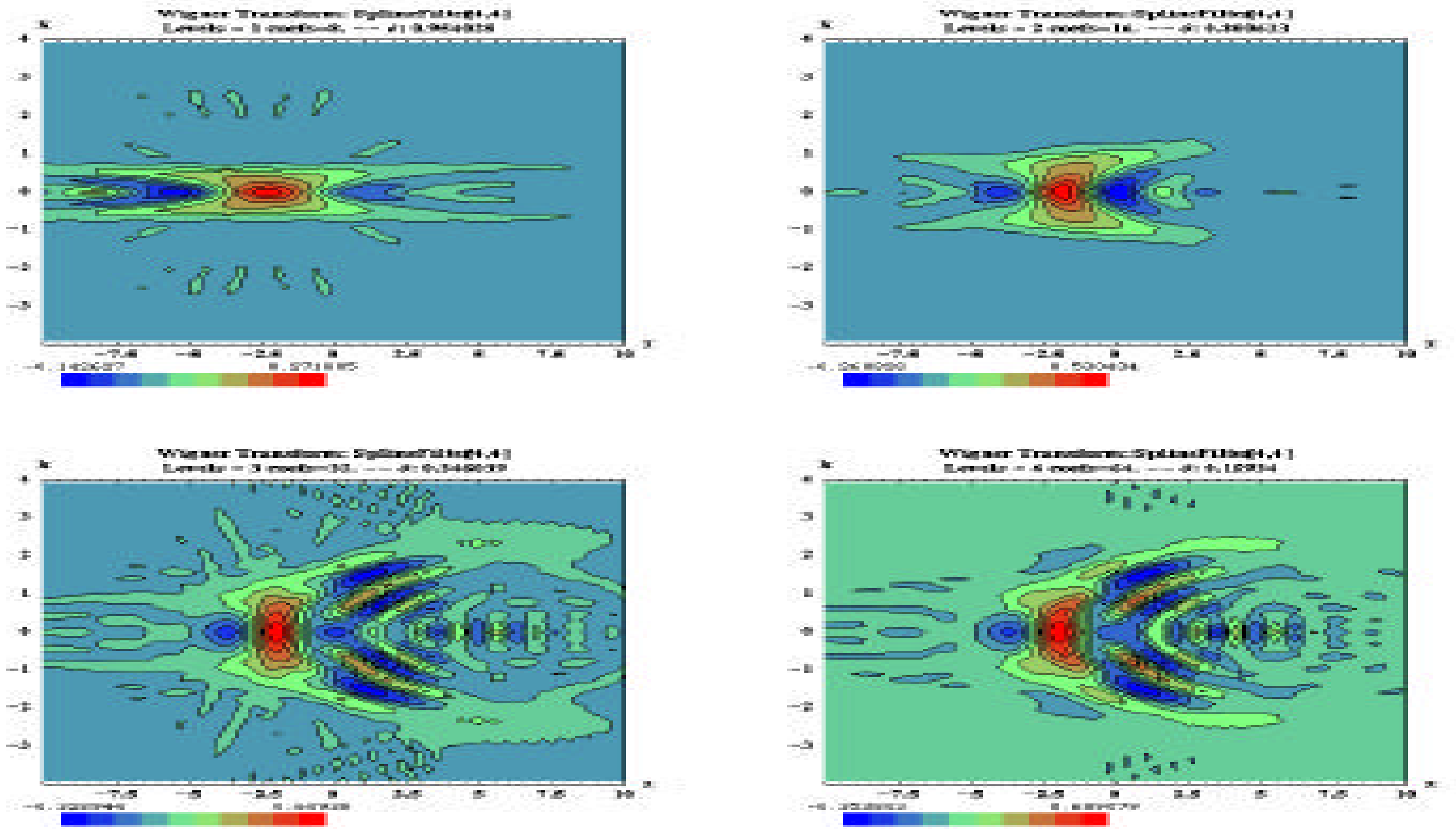


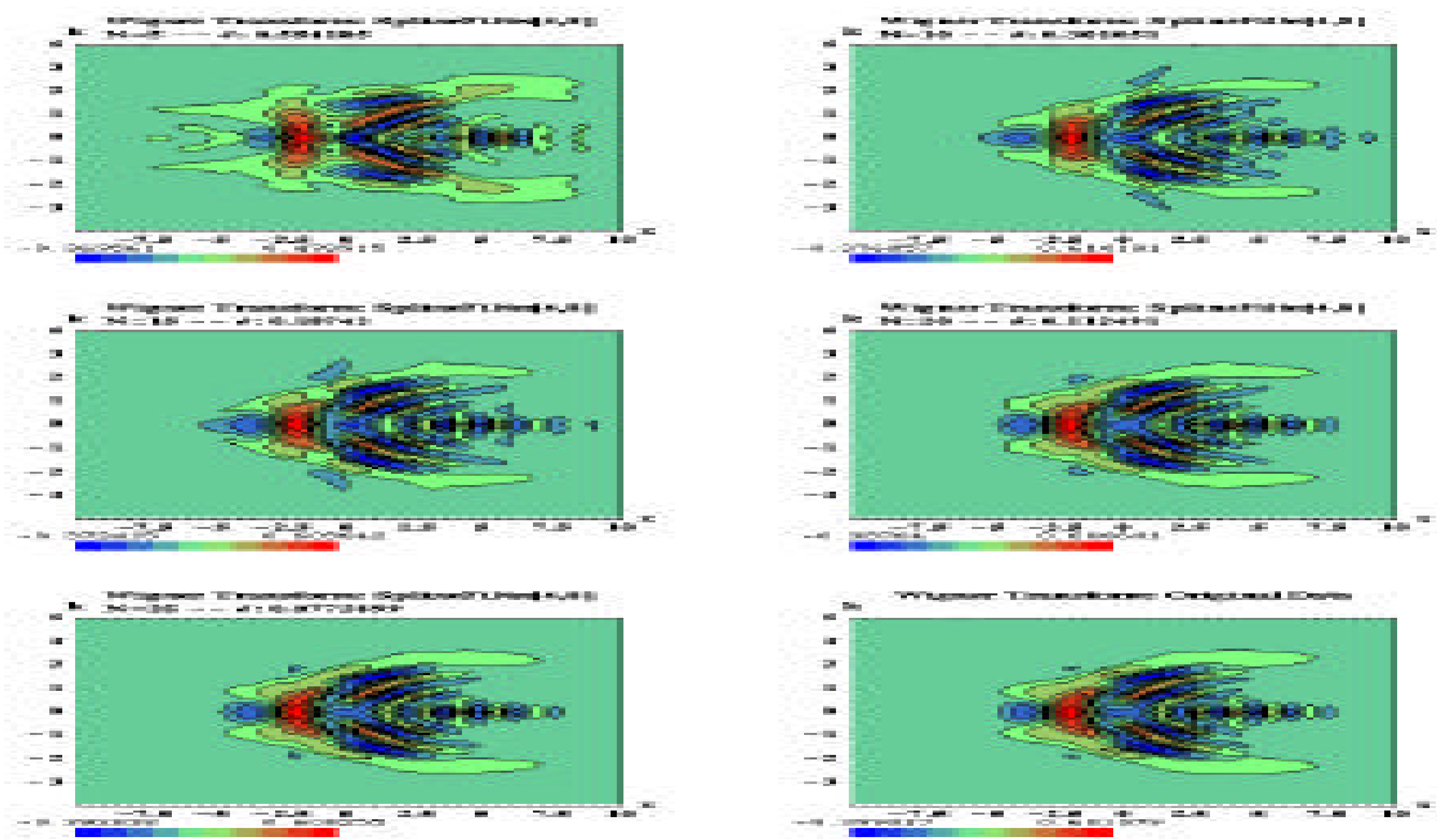
Level by MRD Level WRMR Analysis Using Quartic Spline [4,4]

(L1/95%, L2/80%, L3/35%, L4/16%)



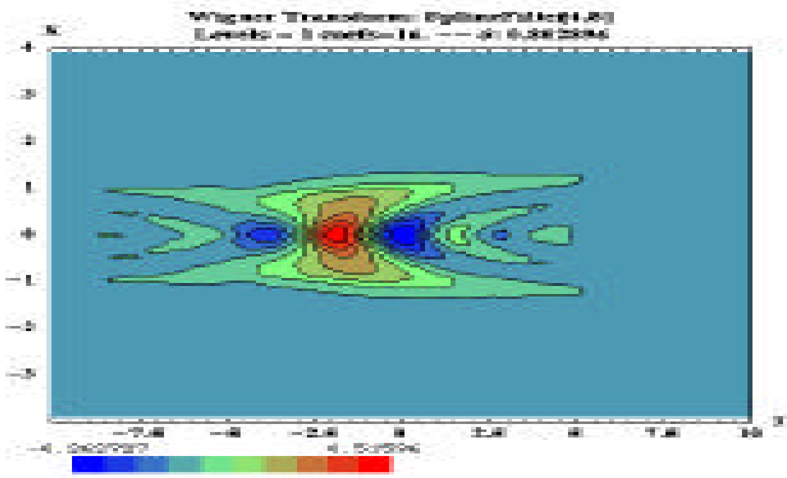
Mean Square Error in Spline[4,8] MRD's Performance in Wigner Land as We Add Largest Coeffs (5/55%, 10/30%,15/19%, 20/11%, 25/7.7%)

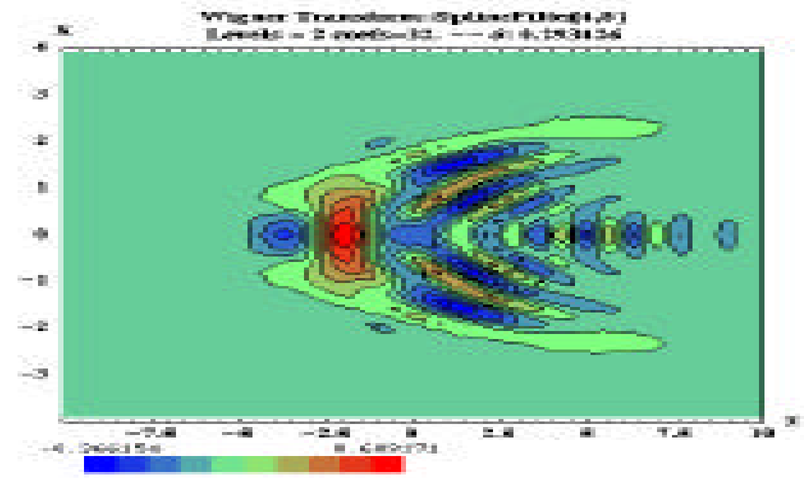
Polymath Research Inc.

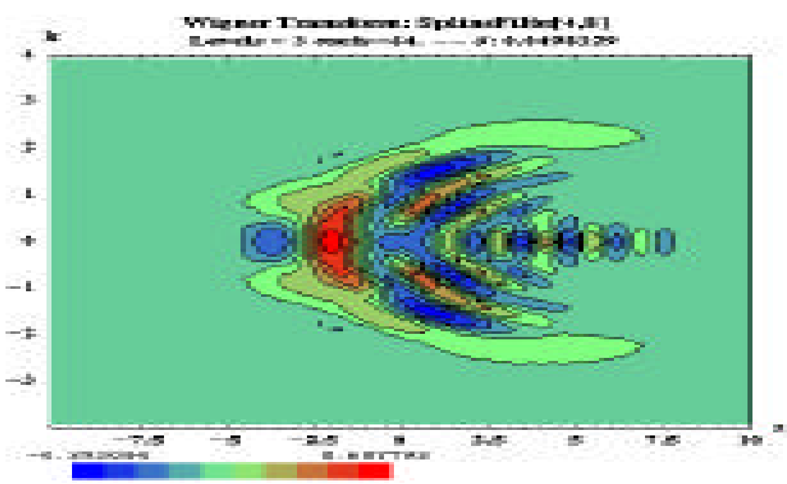


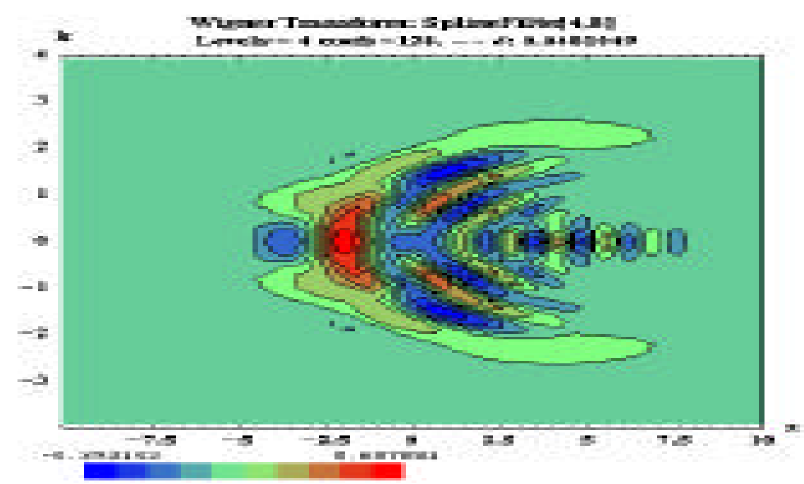
Level by MRD Level WRMR Analysis Using Quartic Spline [4,8]

(L1/80%, L2/29%, L3/4.98%, L4/4.8%)









Wigner Function Representation of the MRD Using Shannon 4

(5/77%, 10/65%, 15/57%, 20/51%, 25/50%)

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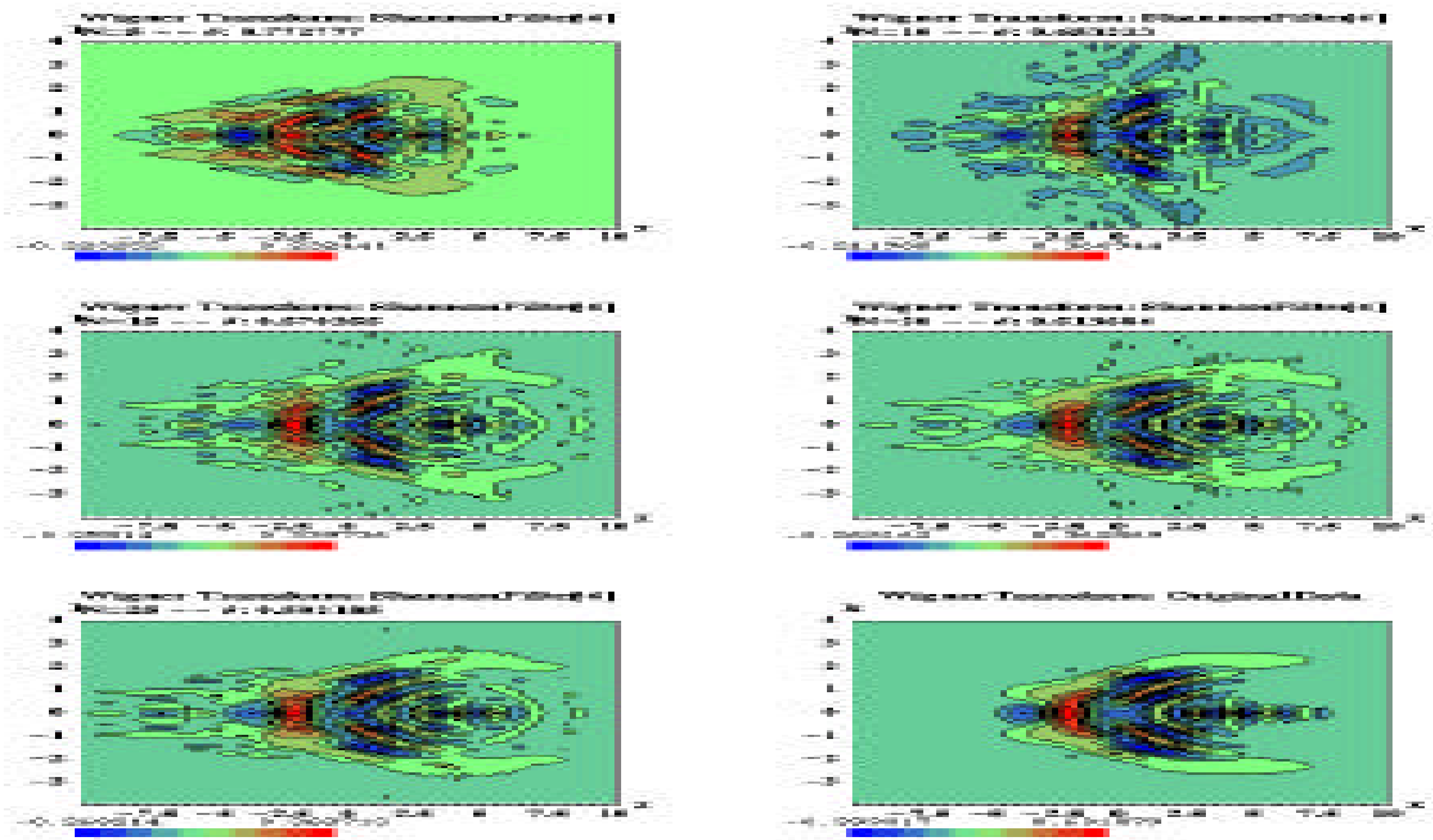
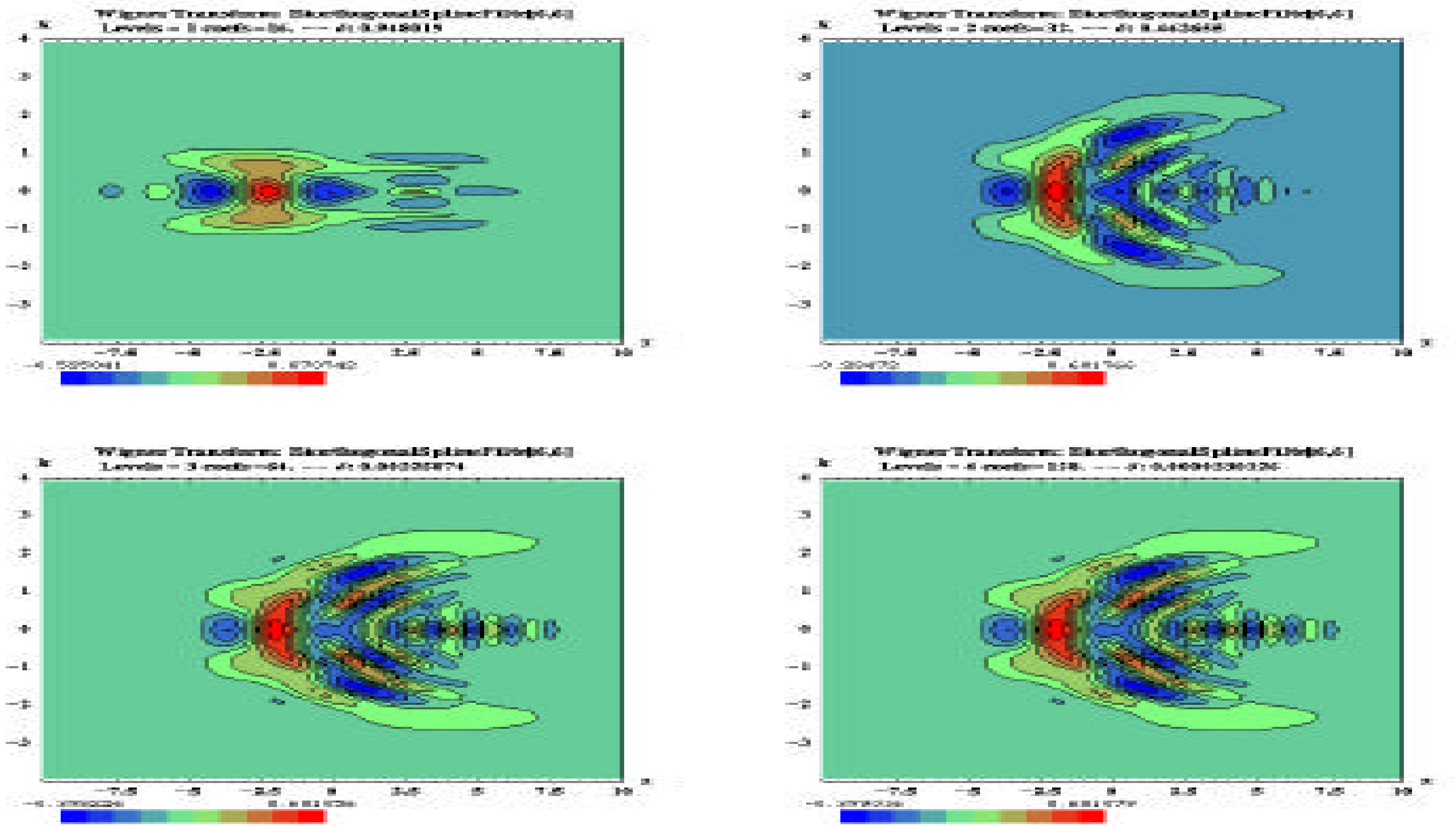


Level by MRD Level WRMR Analysis Using Biorthogonal Spline [6,6] (L1/95%, L2/46%, L3/0.2%, L4/0.003%)



Haar Wavelet's Wigner Transform In Phase Space

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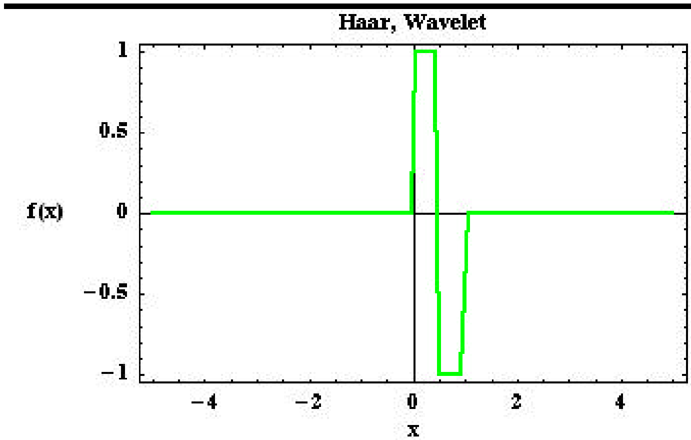

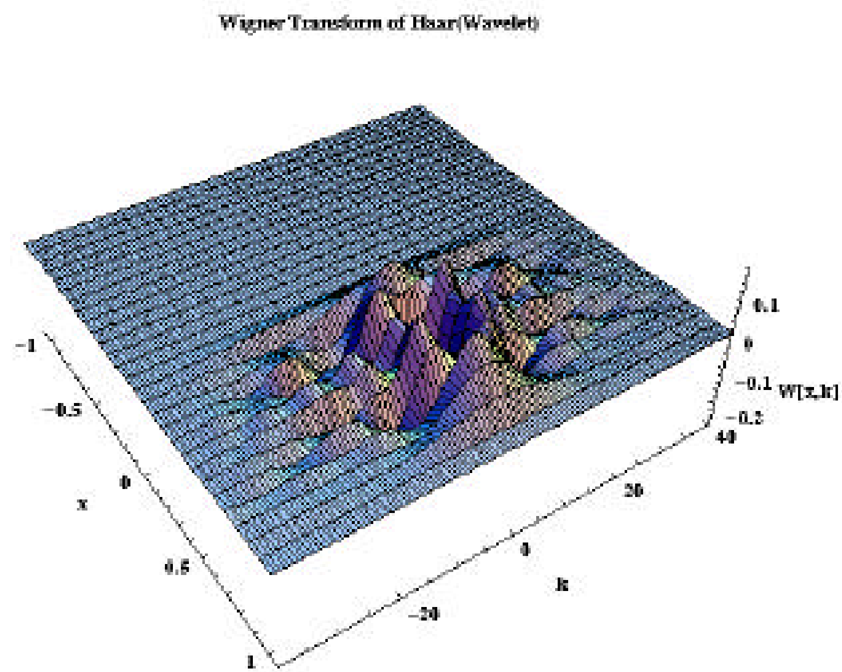

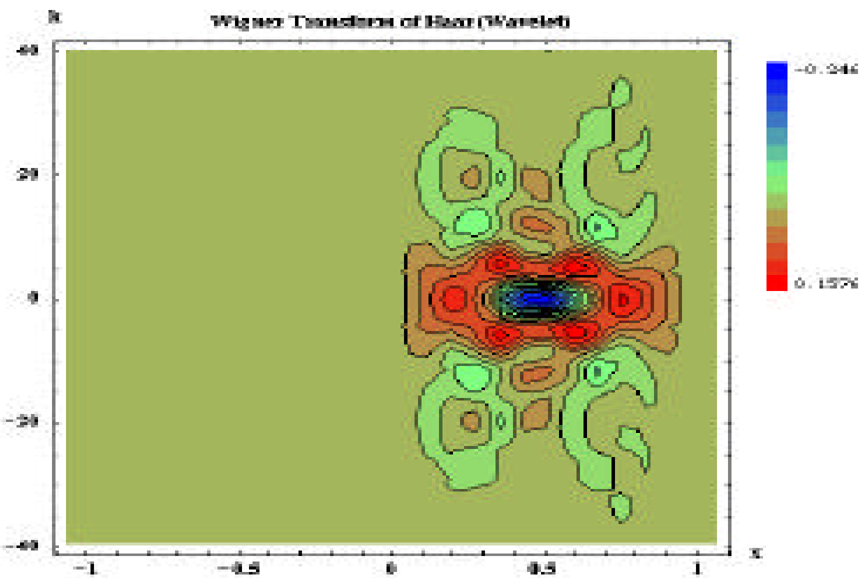

Daubechies 3 Wavelet's Wigner Transform In Phase Space

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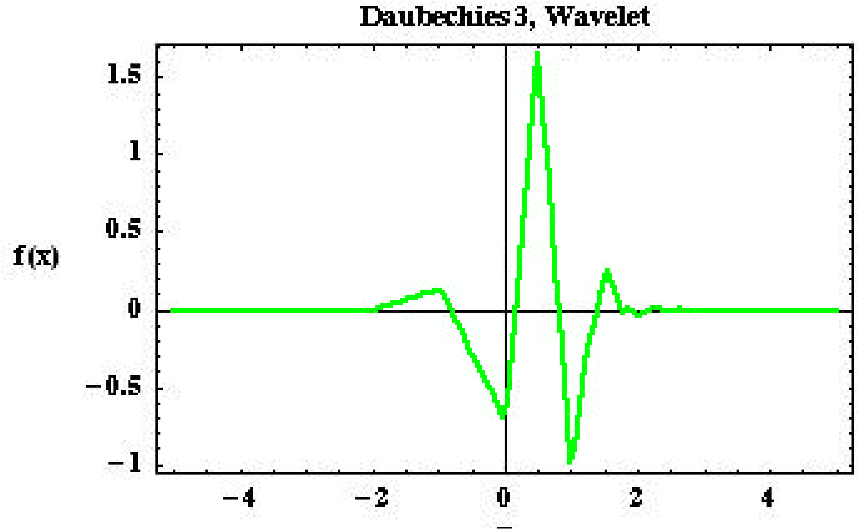
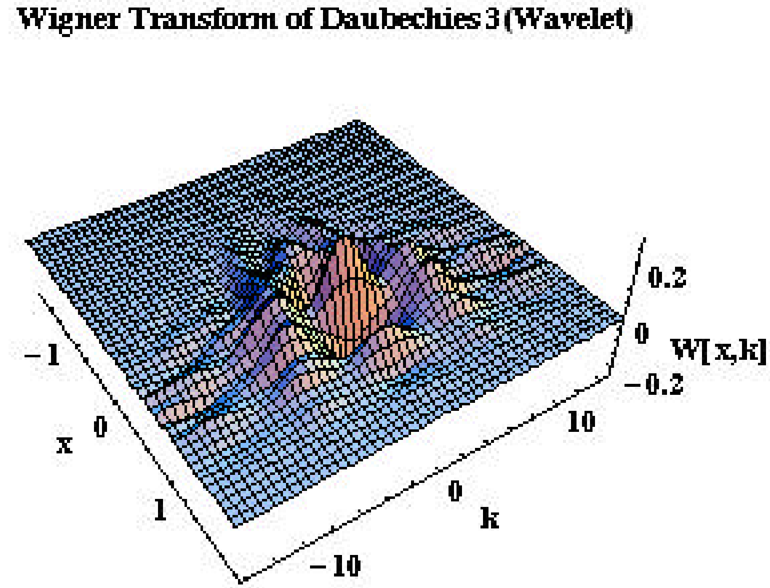
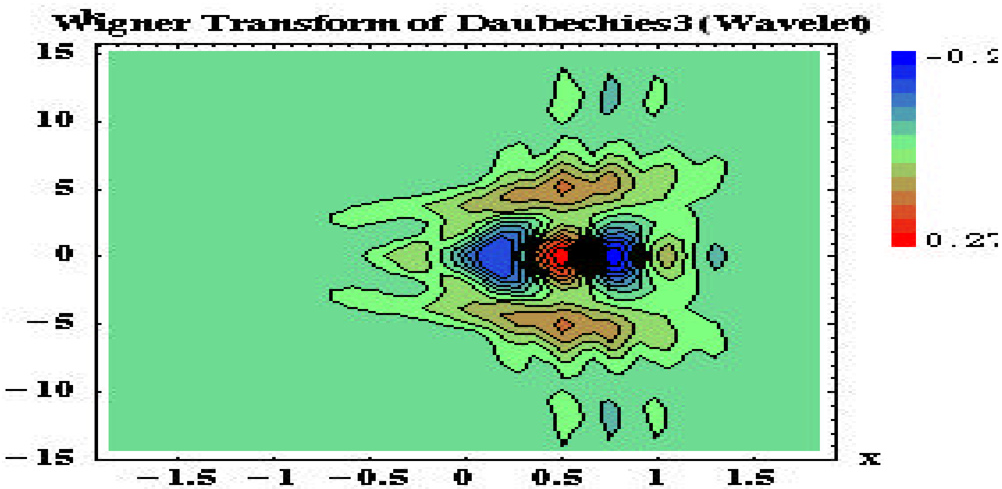
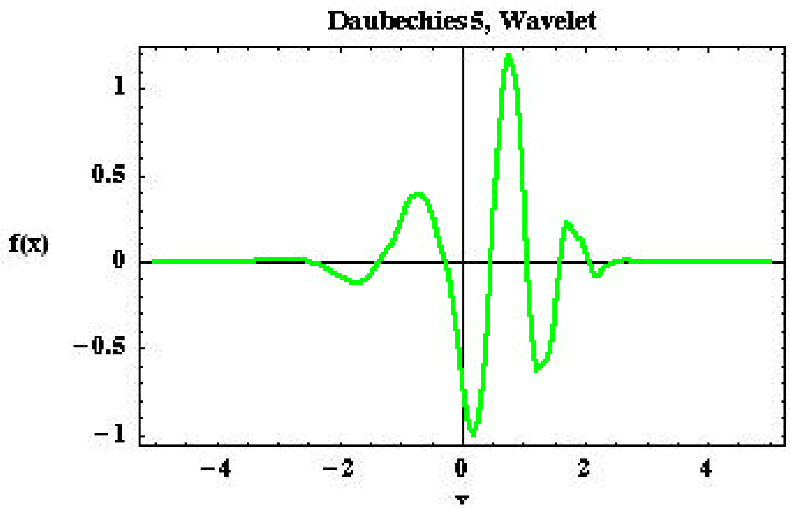
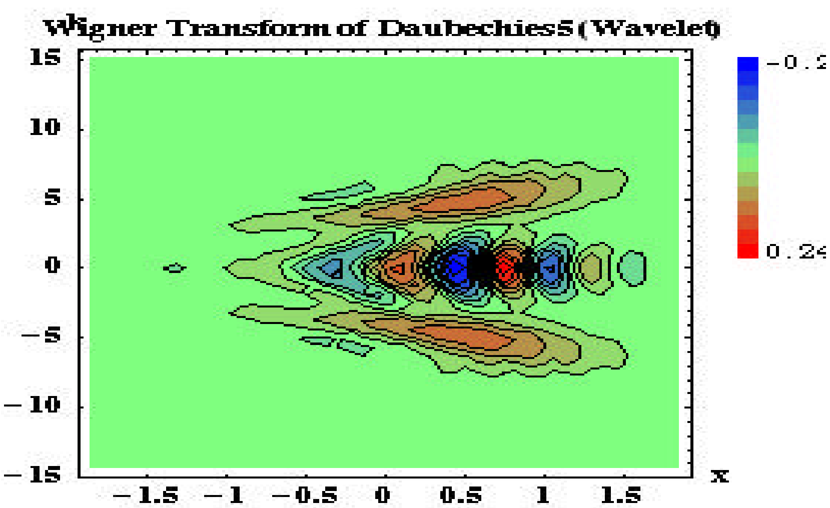


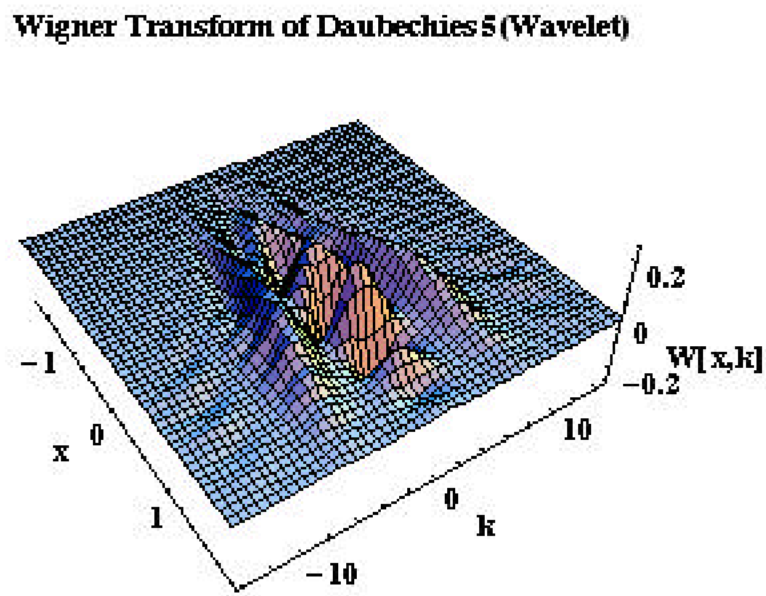




Daubechies 5 Wavelet's Wigner Representation In Phase Space







Daubechies 6 Wavelet's Wigner Representation In Phase Space

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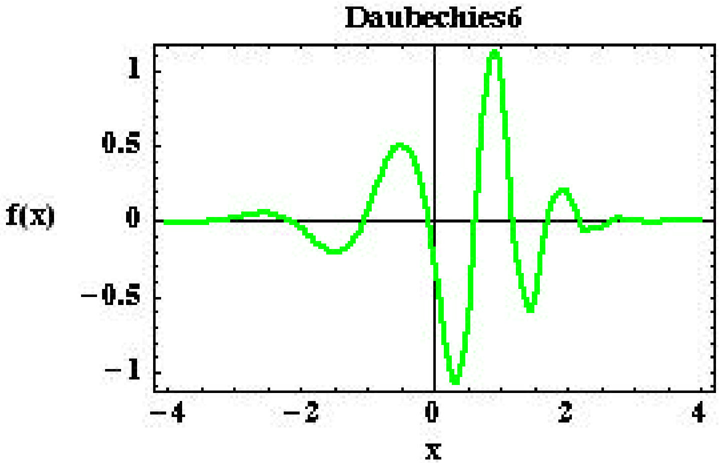
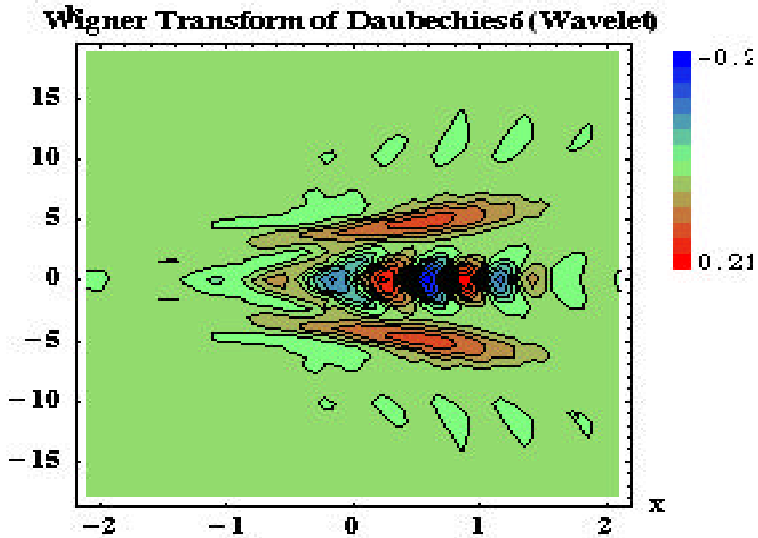
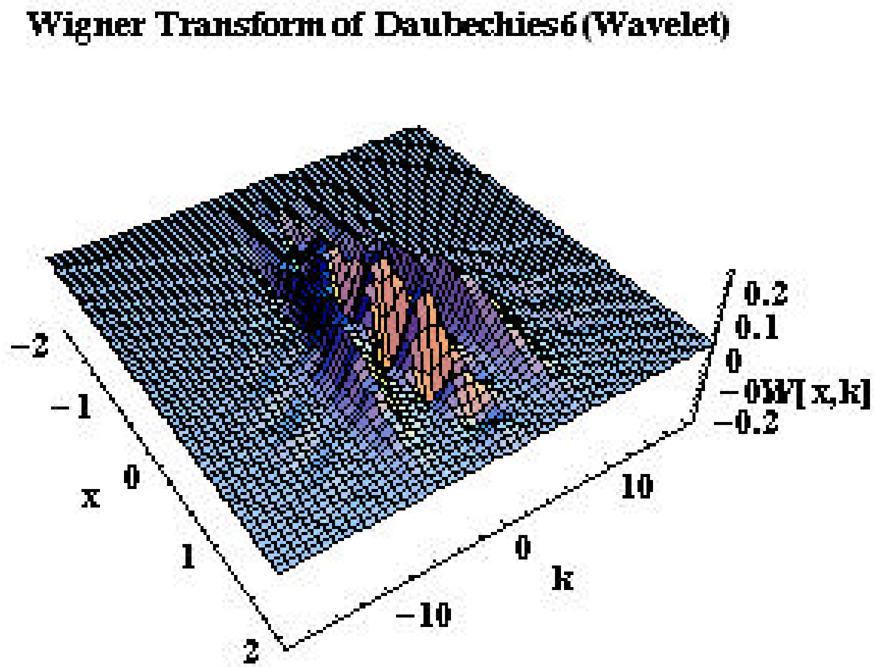
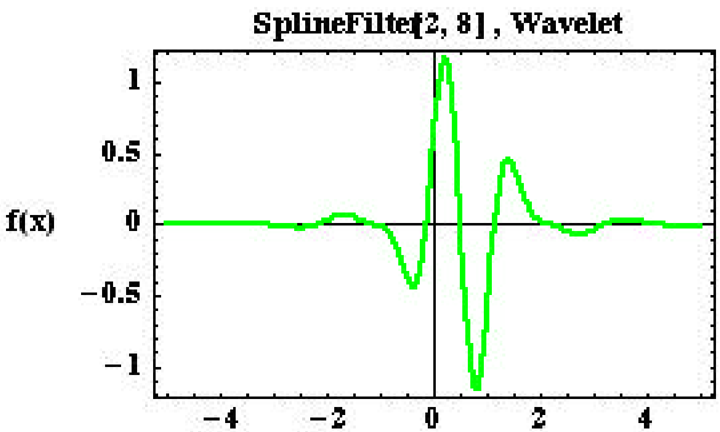
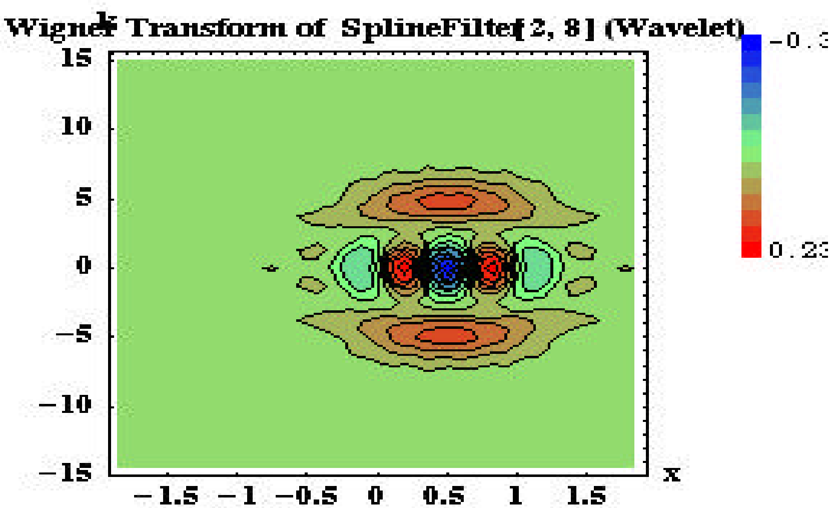


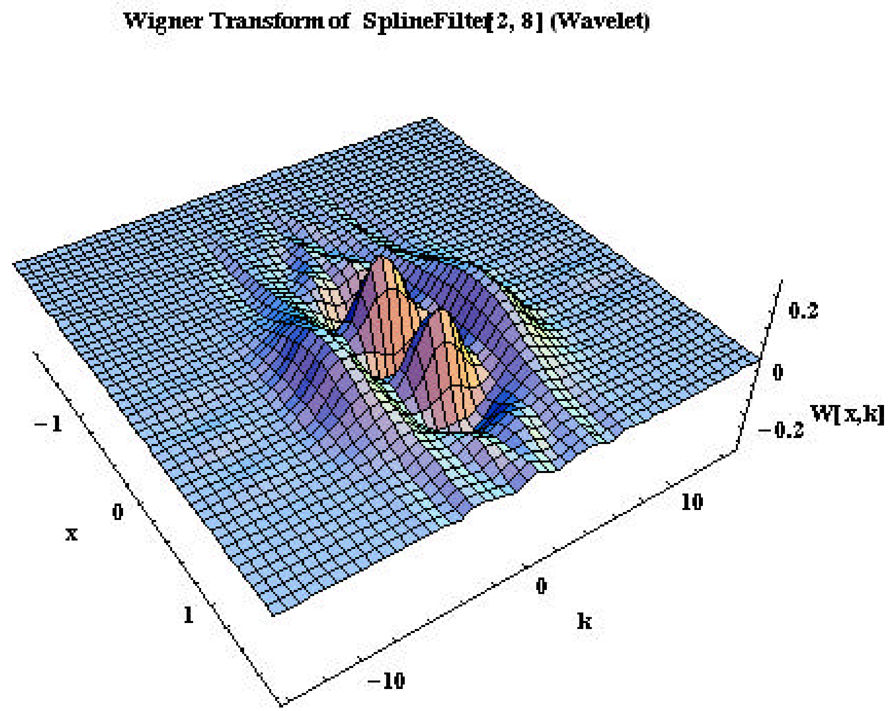




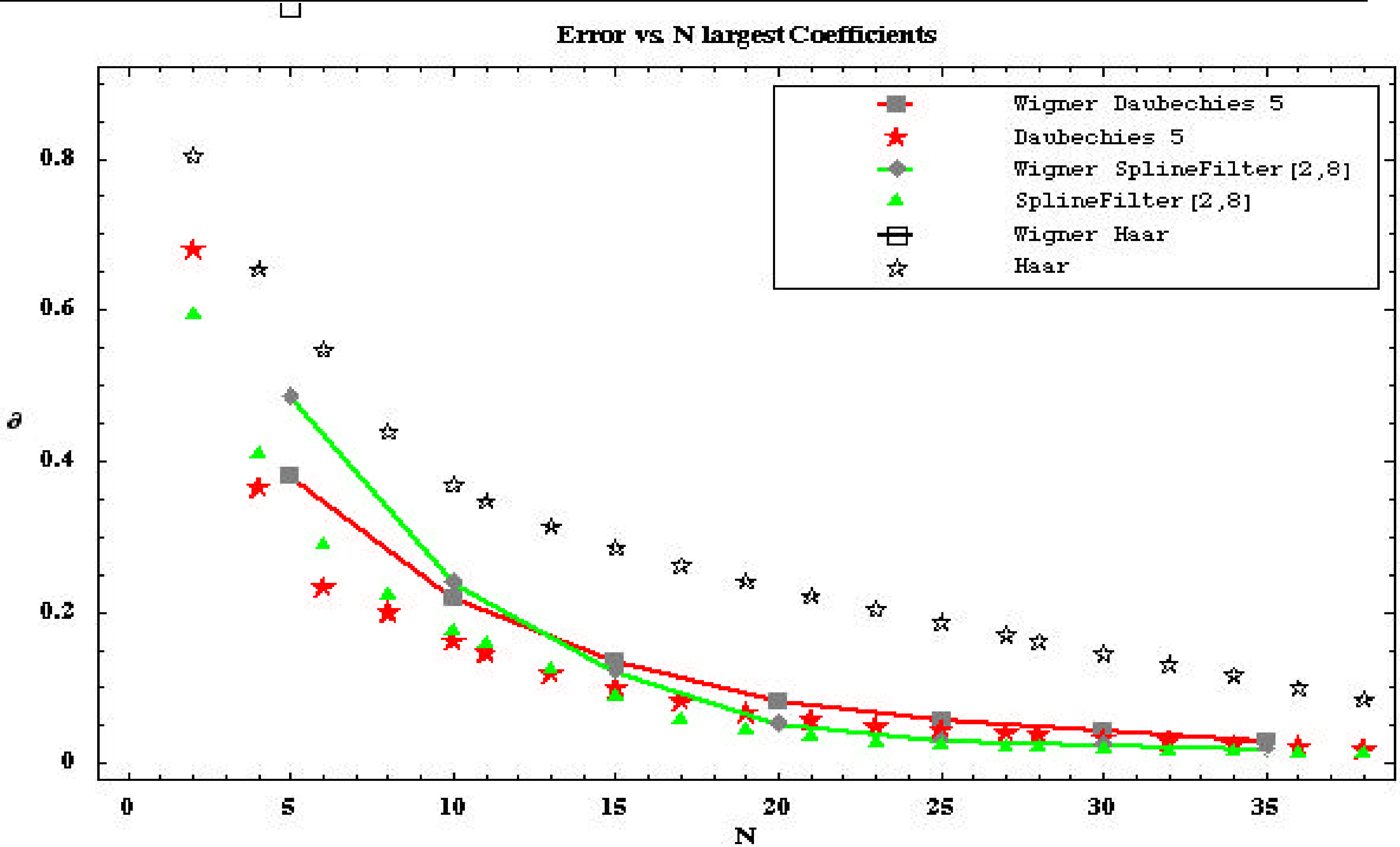
Quadratic Spline Filter Wavelet's Wigner Representation



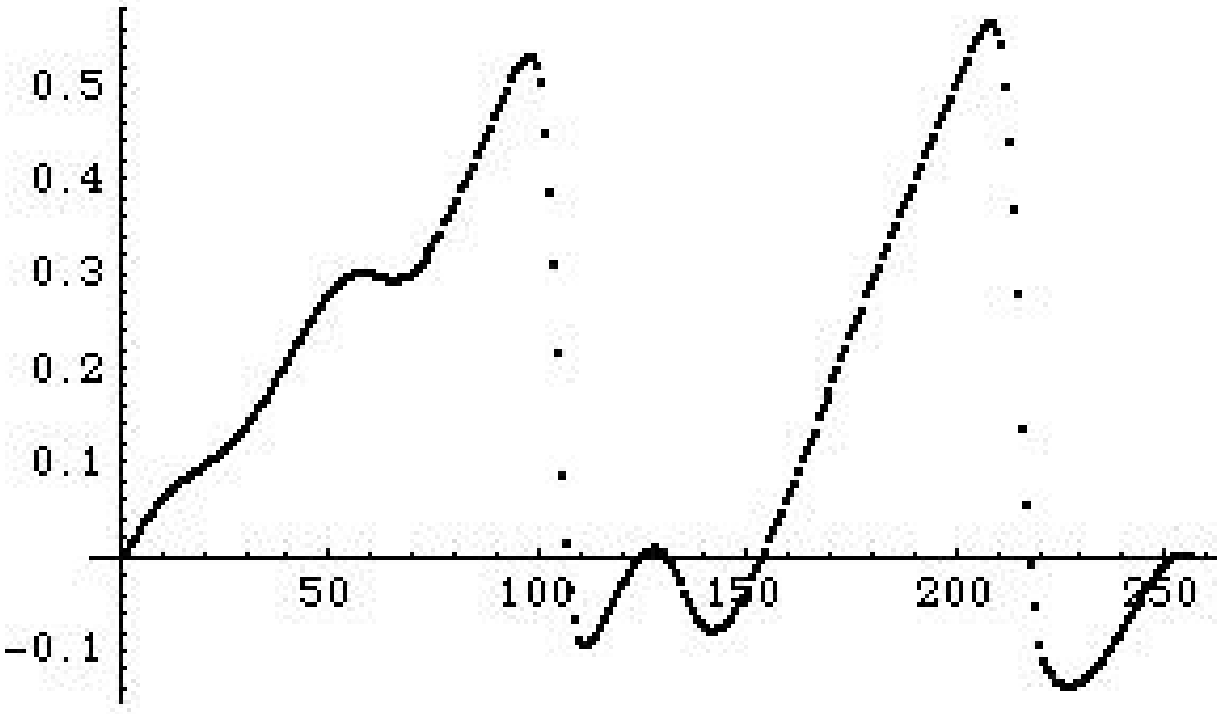


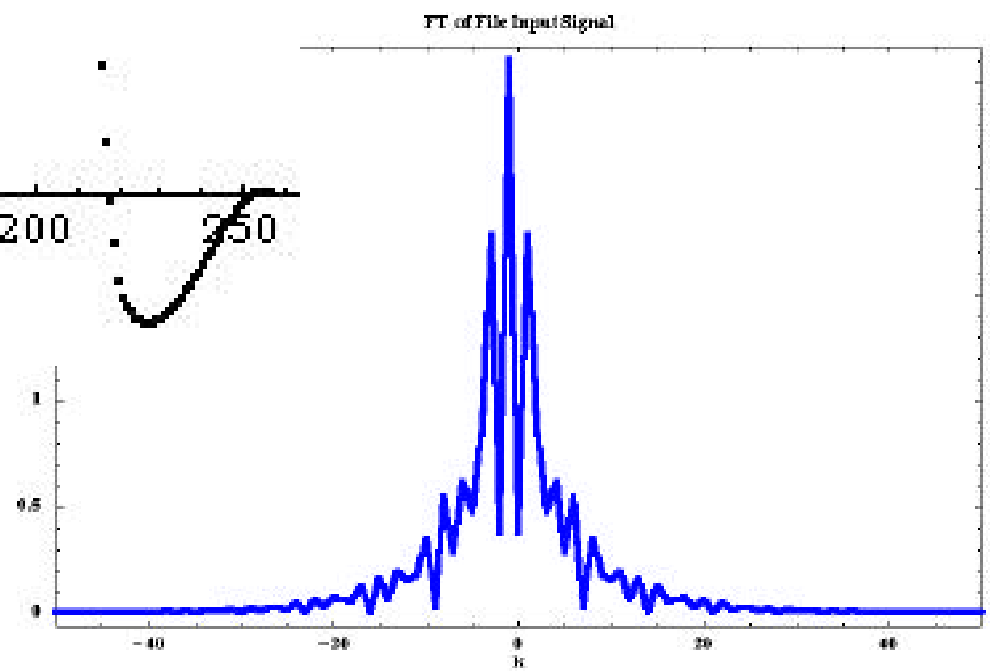


Comparison of Fractional Least Square Error in f(x) vs $W_f(x,k)$



Sample Portion of a Solution to the Driven Burger's Equation: Random Superpositions of Viscous Shocks





Relative Sized of the Largest Coefficients of 4 WLT MRDs

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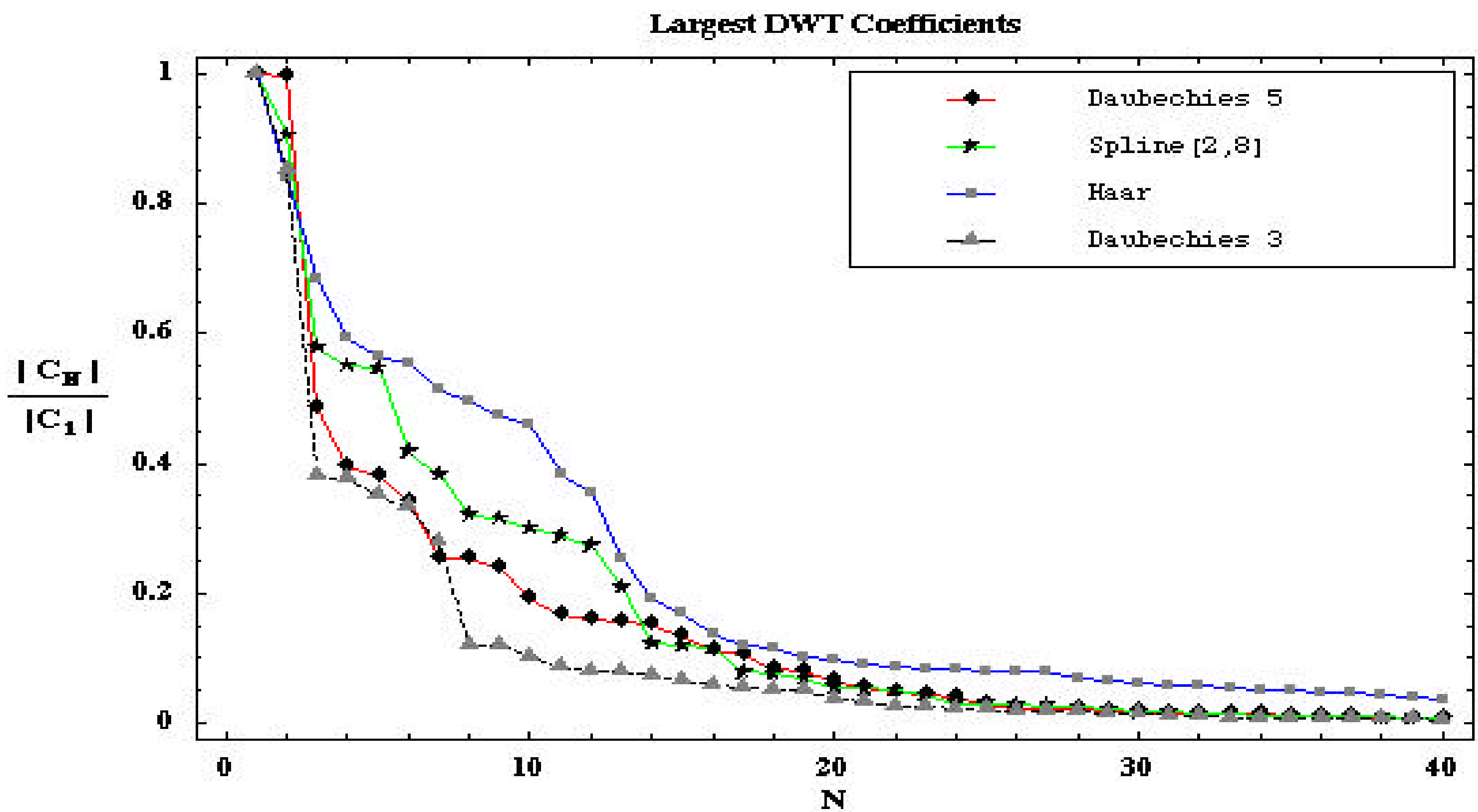

Energy Accumulation Rate in Coefficient Space for 4 WLT MRDs

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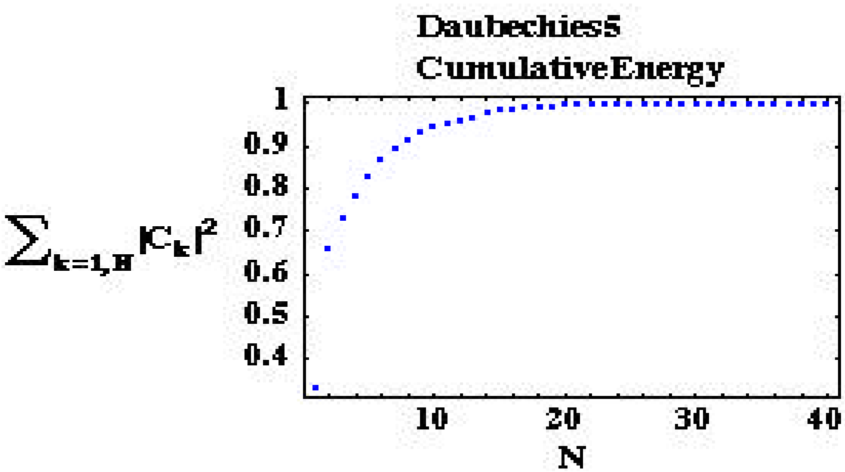
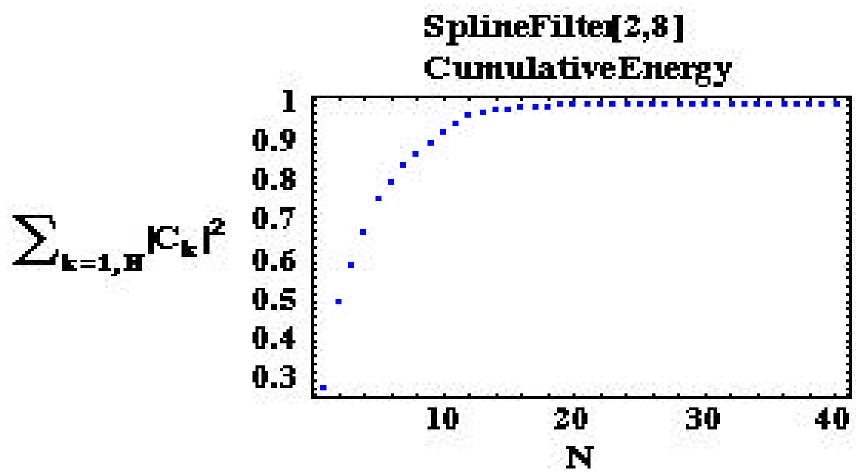
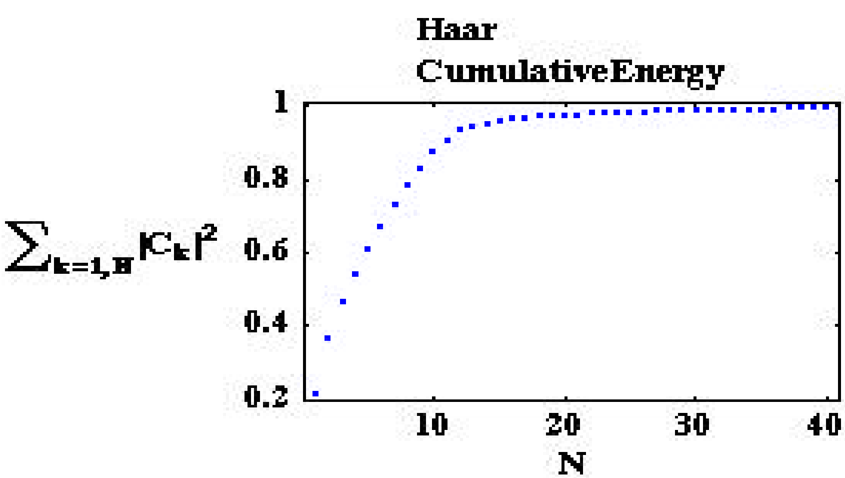
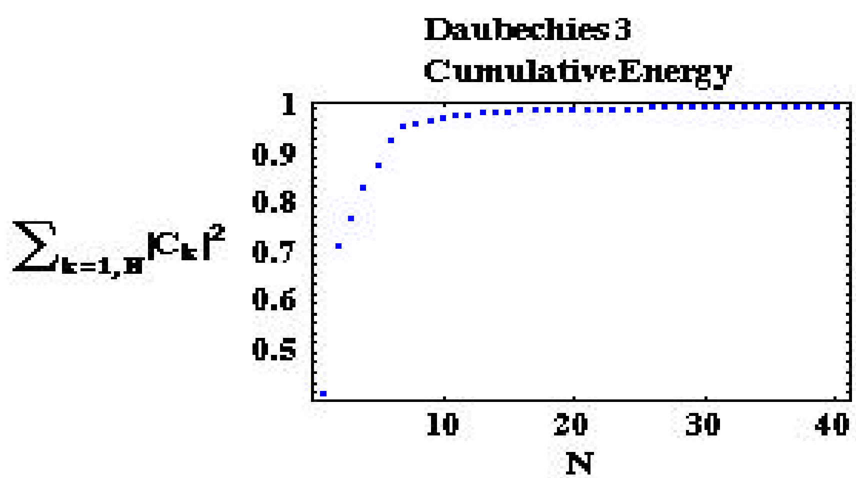
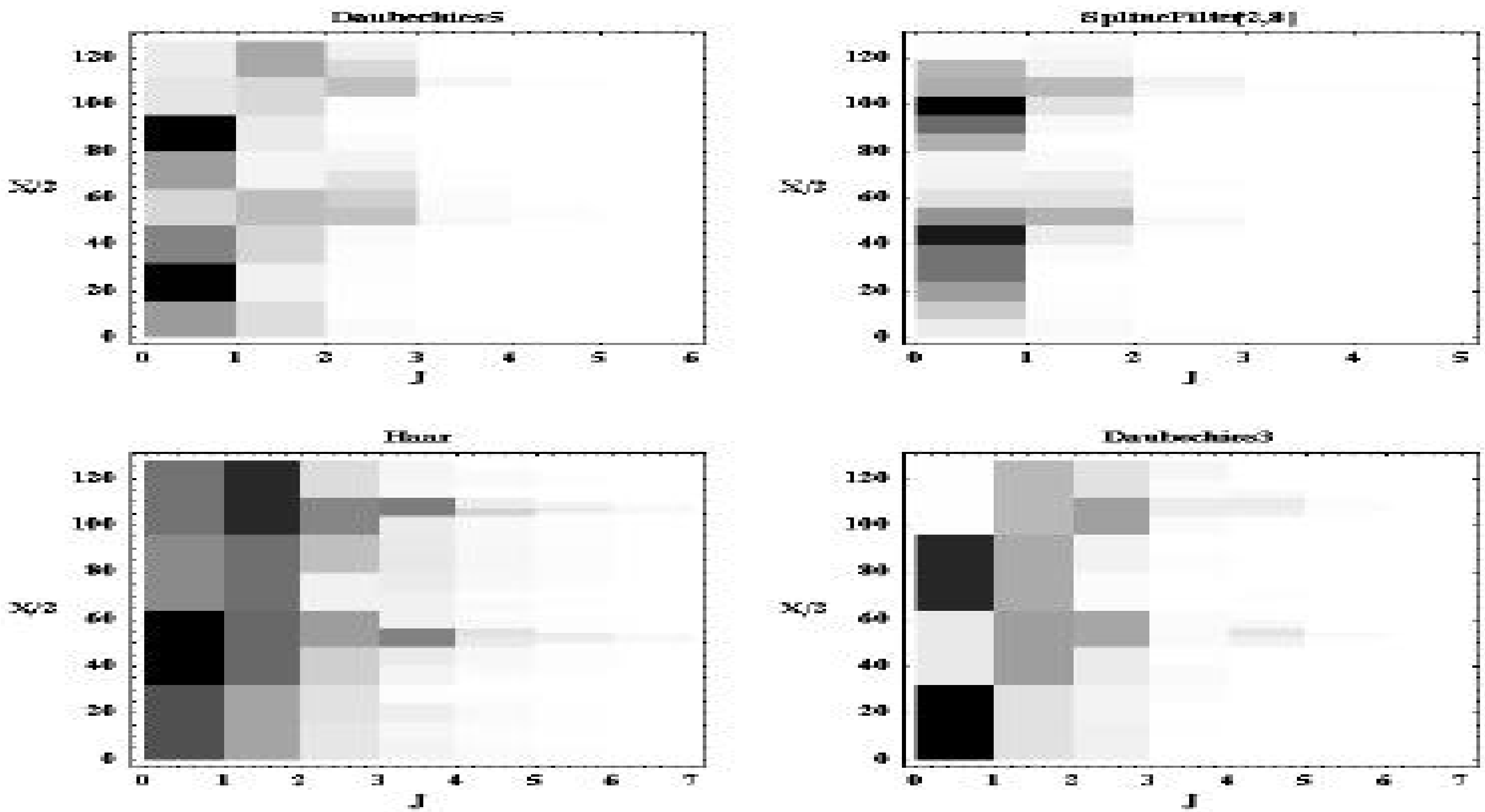








Scalograms of the Random Shock Solution Using 4 WLT MRDs



Fractional Least Square Error In Largest Coefficients Thresholding of the Random Shock Solution

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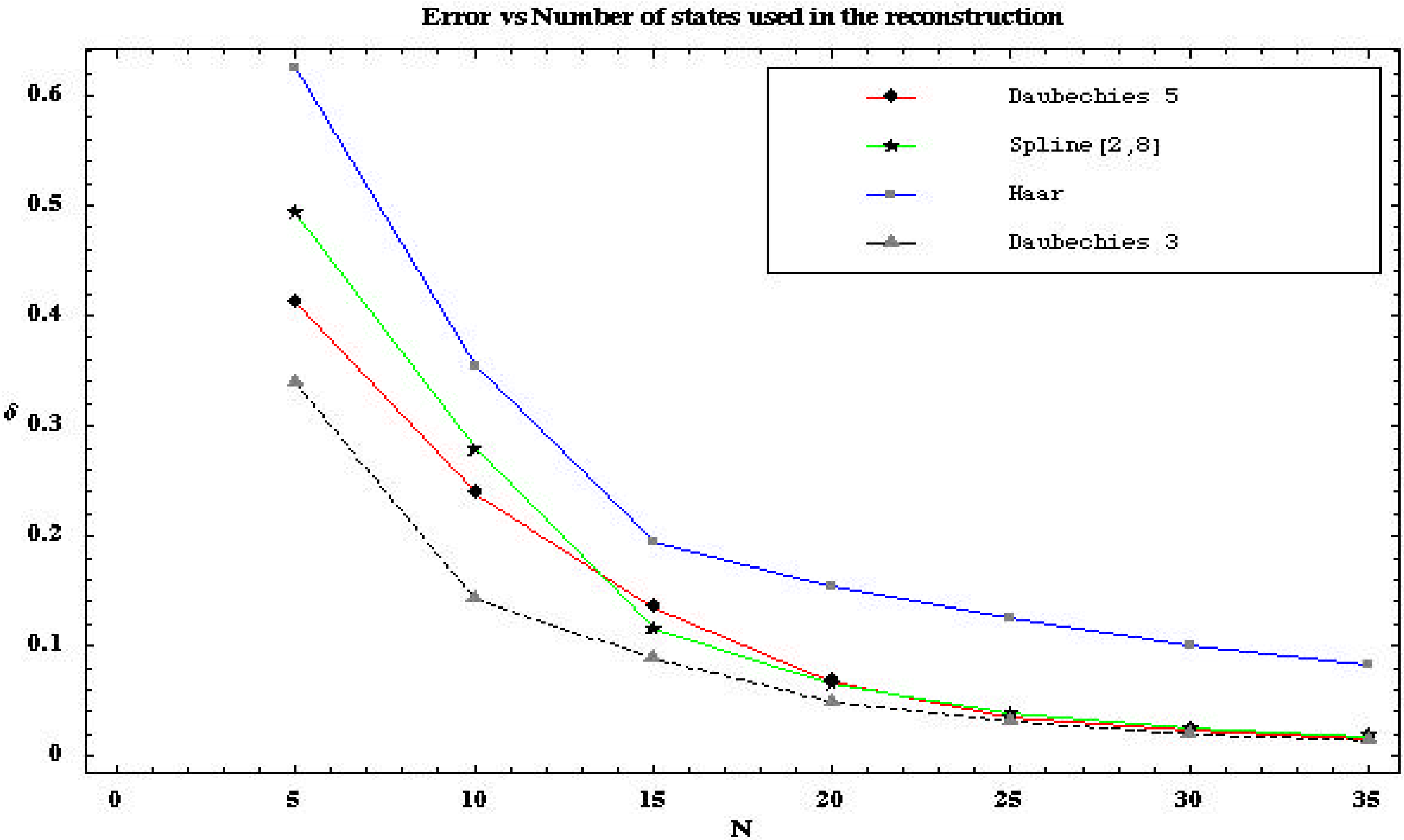

Fractional Least Square Error vs Number of Levels in Reconstruction

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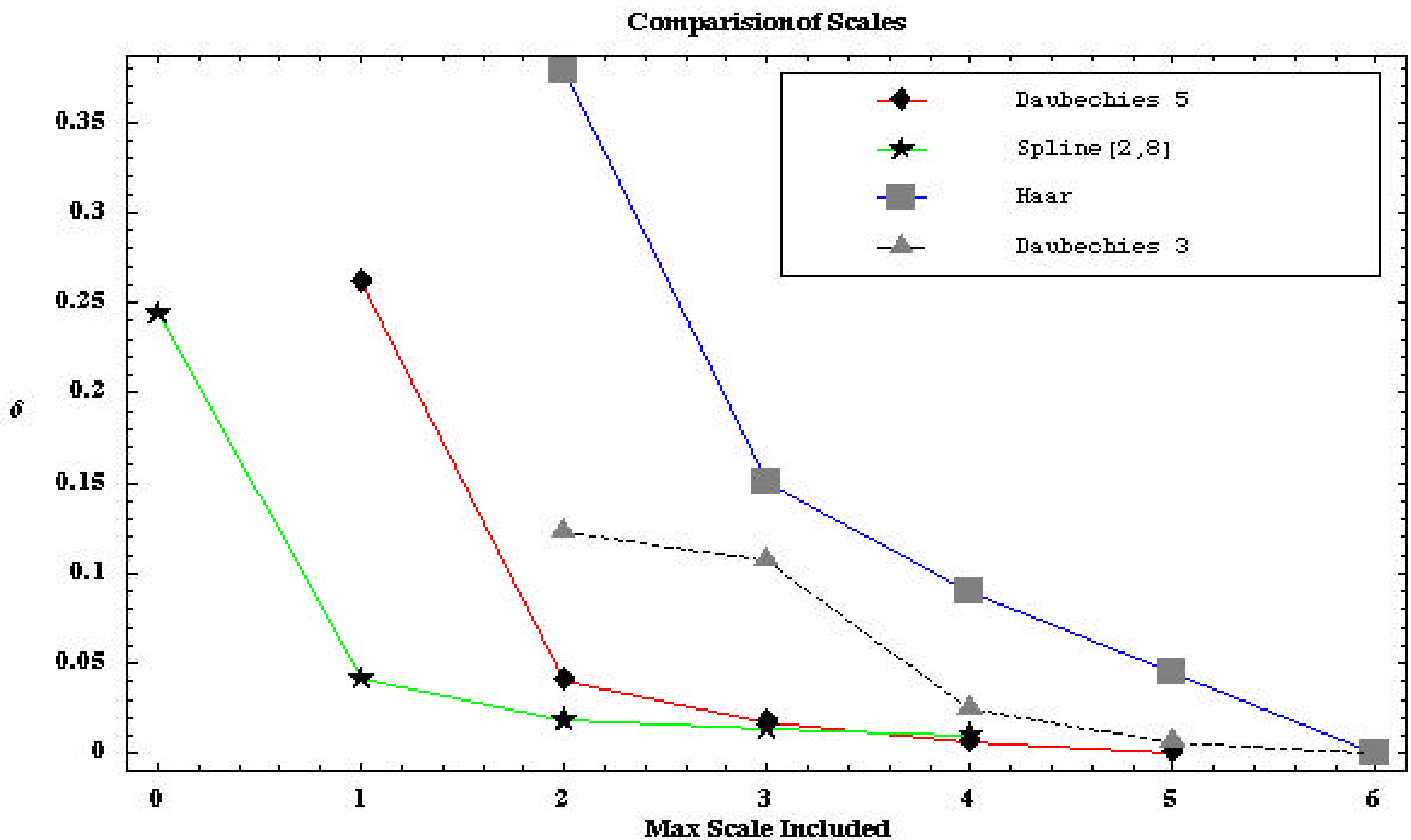
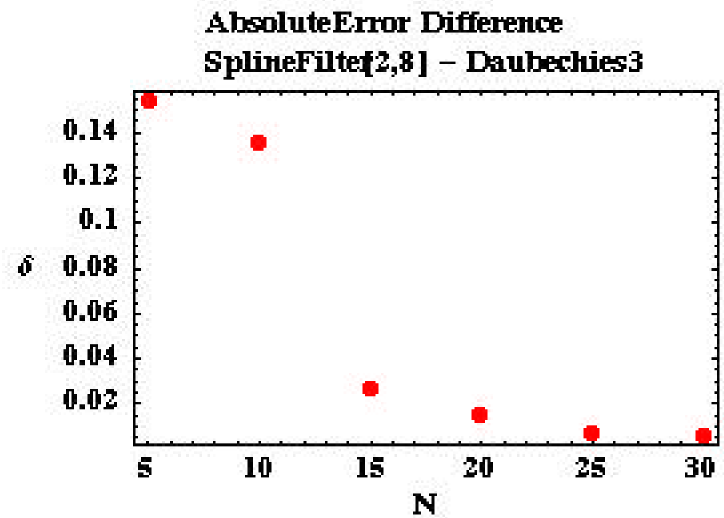
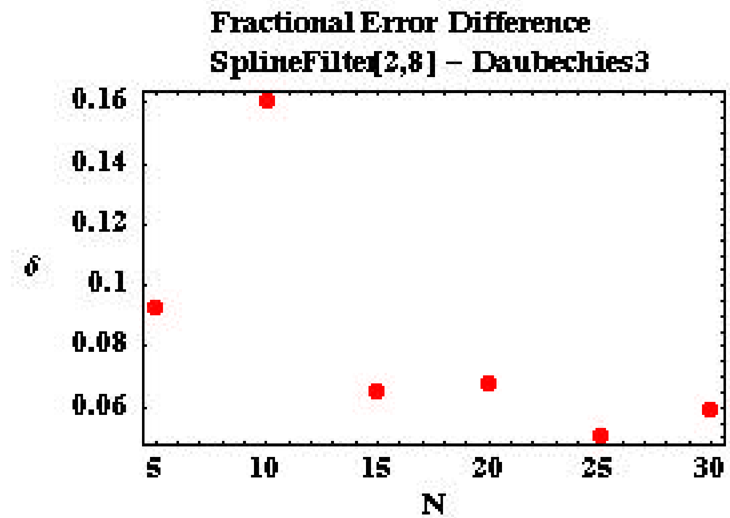
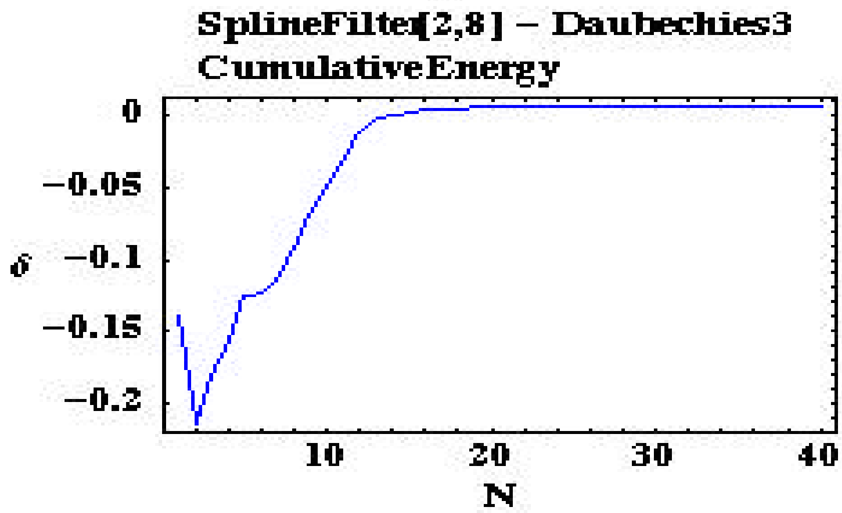
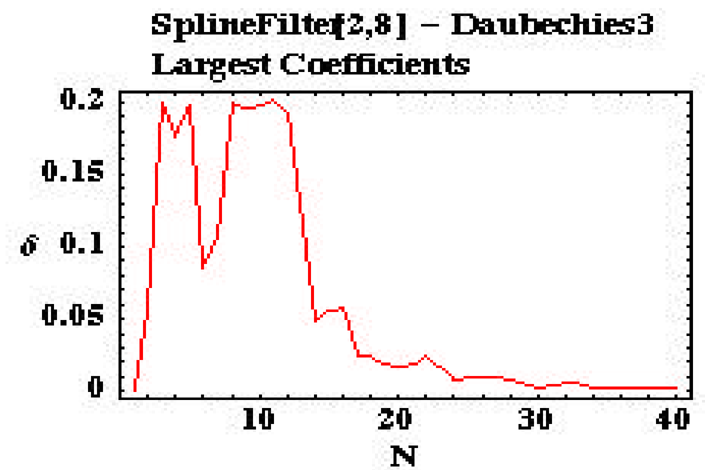


Comparison of Spline[2,8] to Daub 3 in Reconstruction: Daub 3 Wins

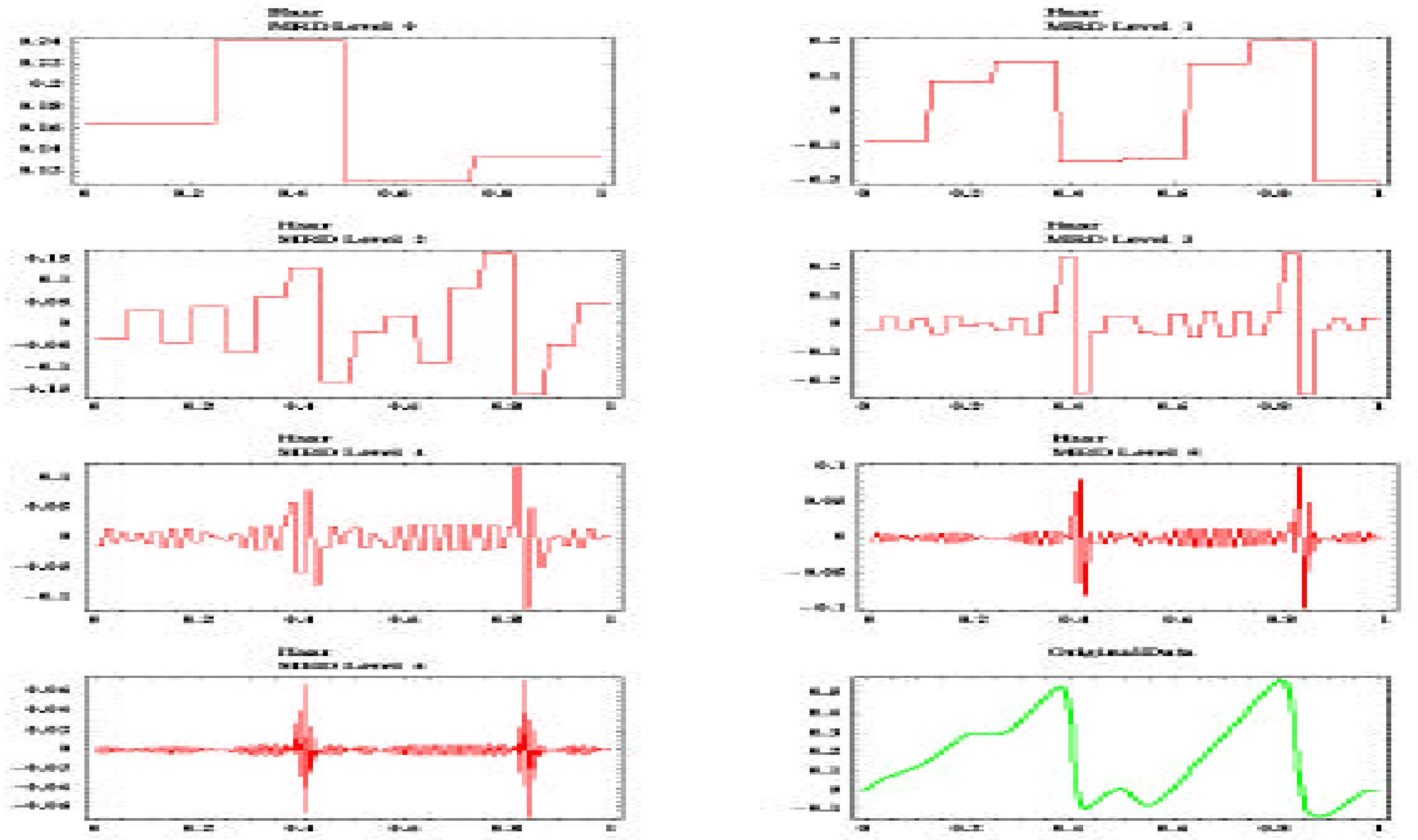








Haar MRD of the Random Shock Solution



Daubechies 3 MRD of the Random Shock Solution

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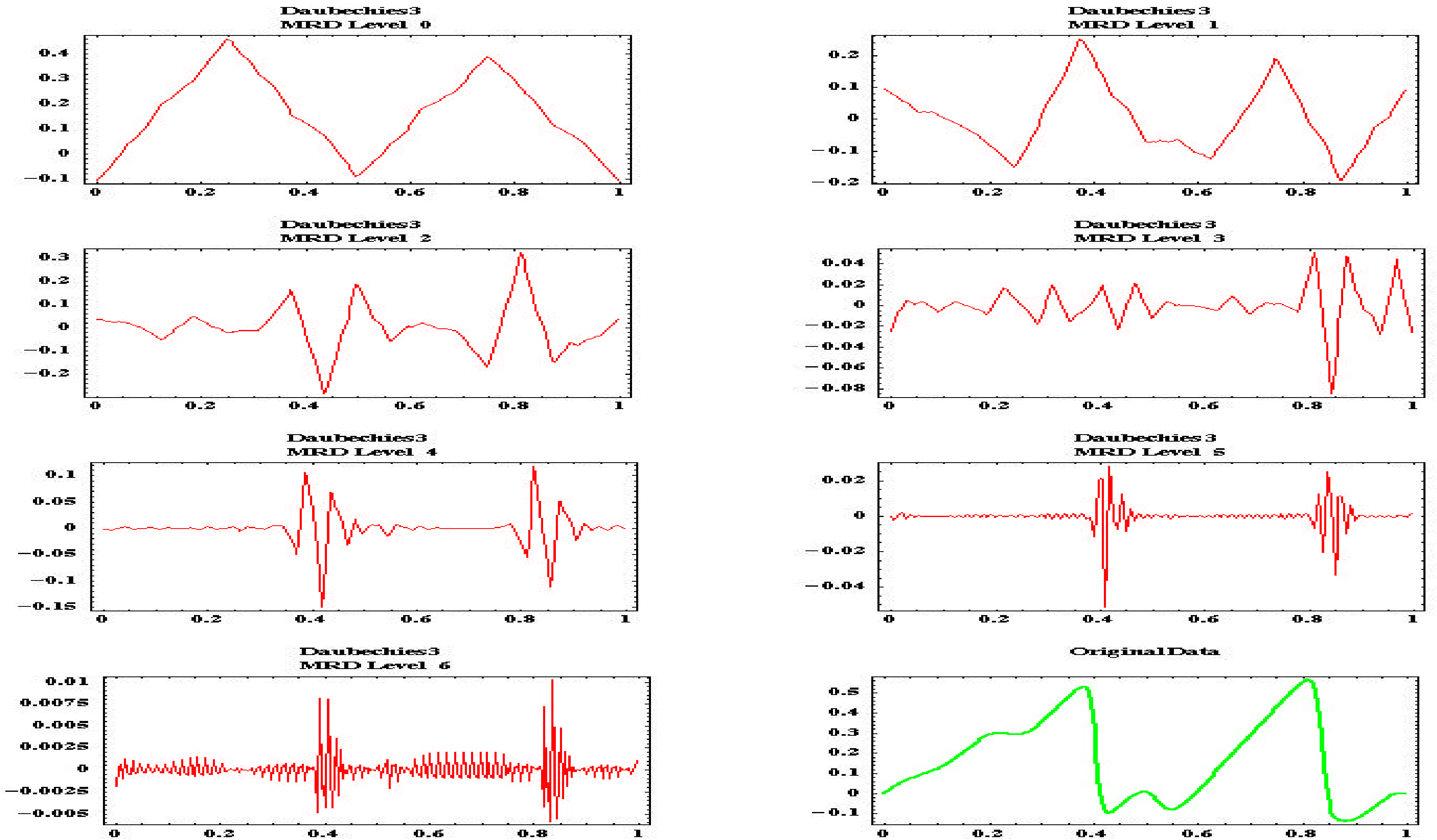
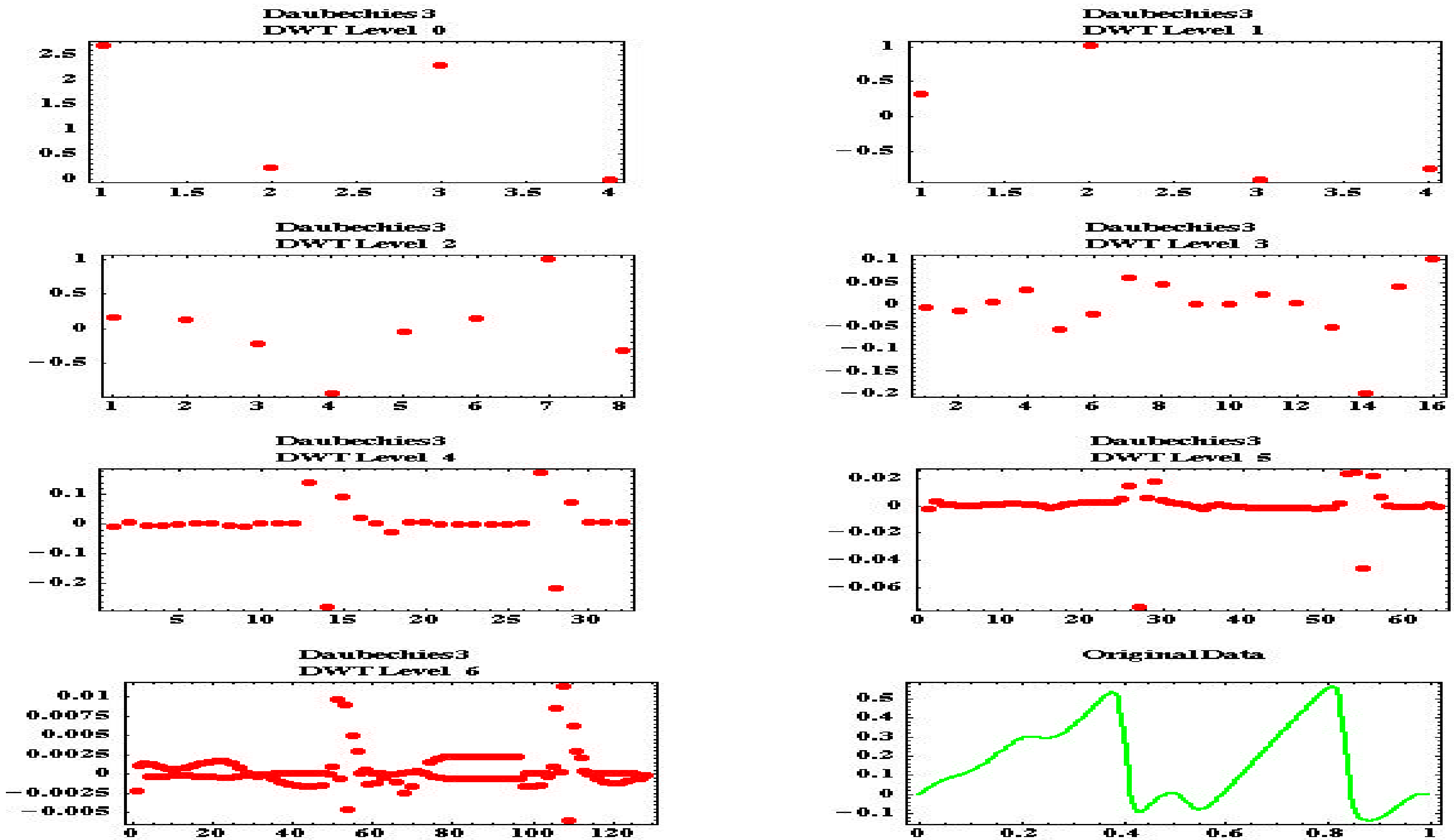
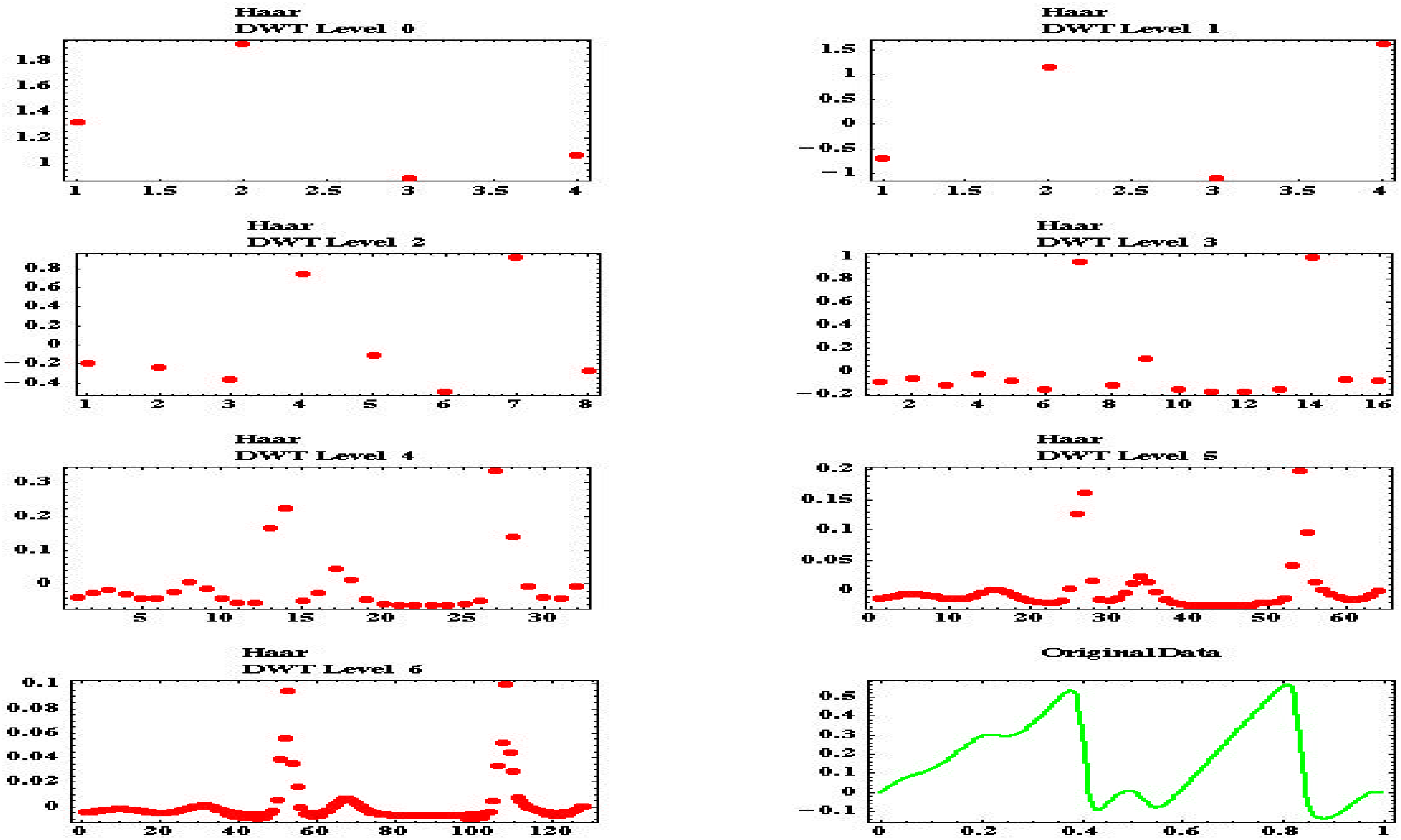


The Coefficients of the MRD Using Daubechies 3



The Coefficients of the MRD Using Haar



Sample Portion of a Solution to the Driven Burger's Equation: Random Superpositions of Viscous Shocks

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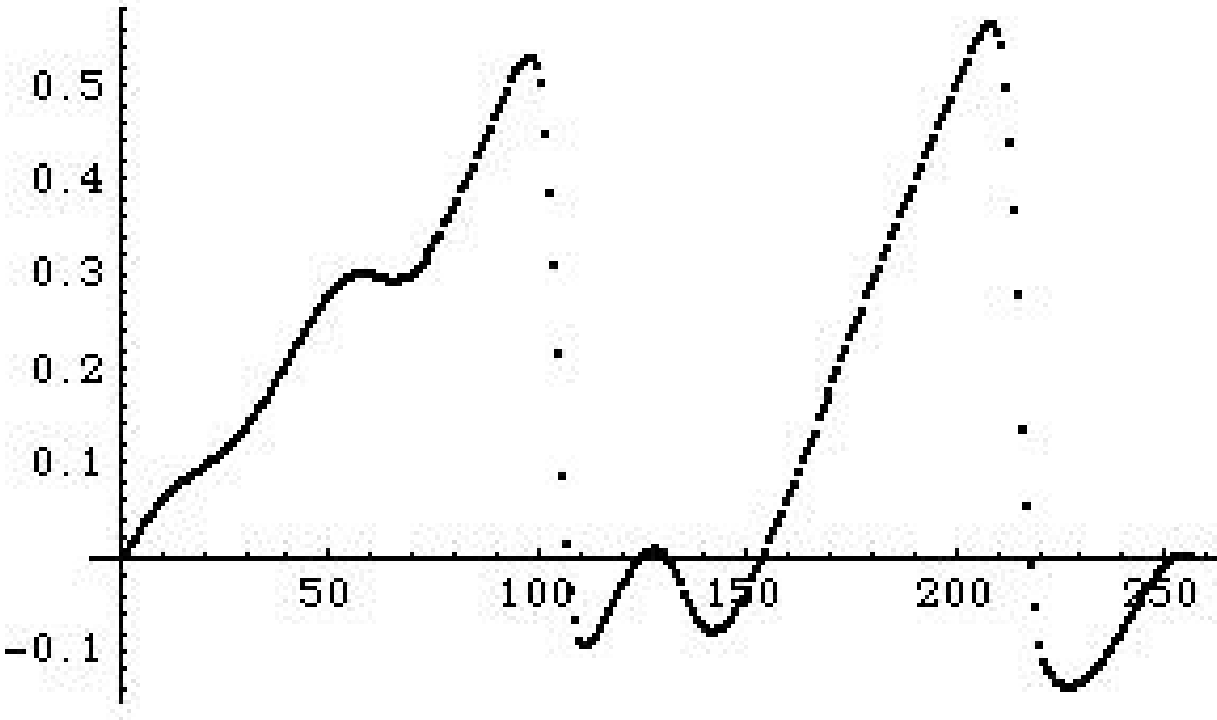
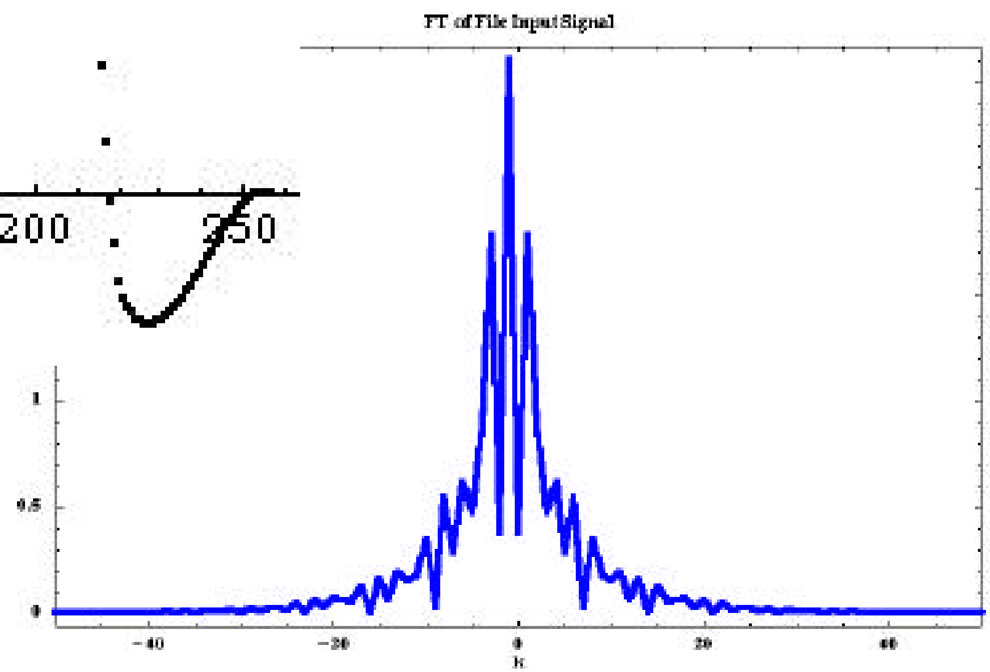
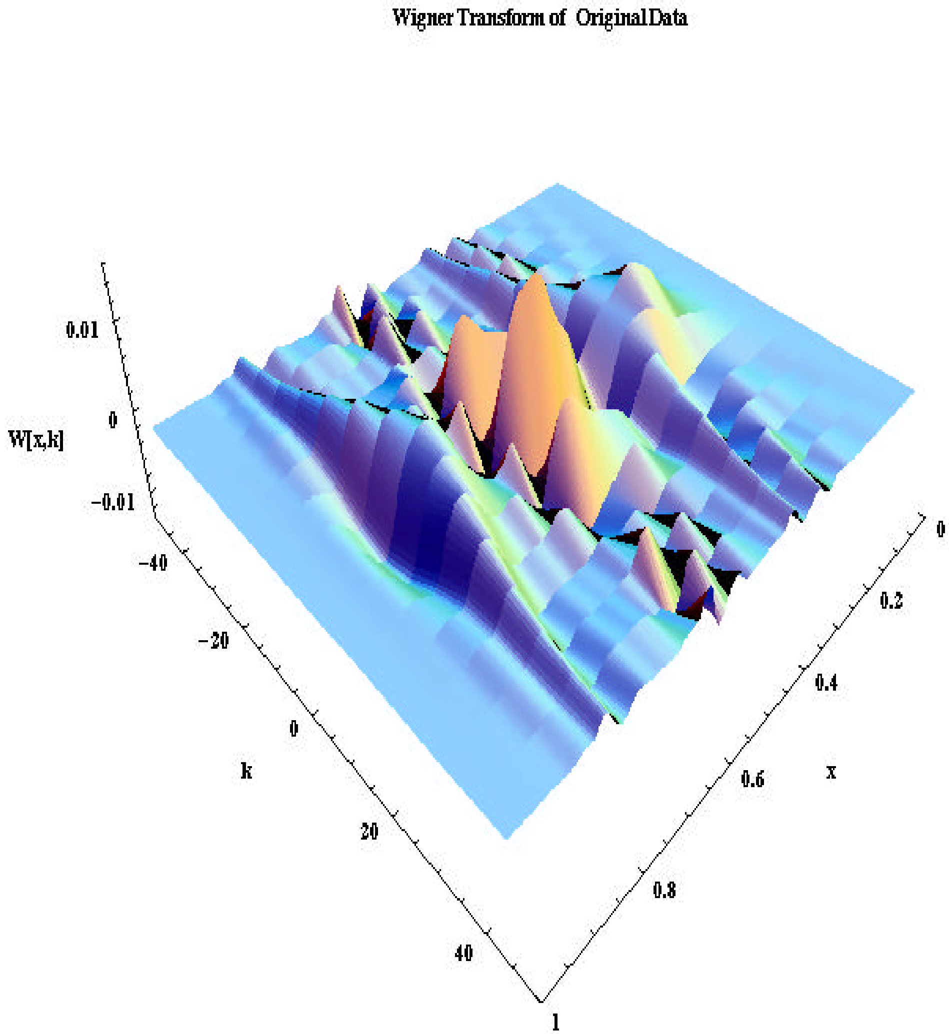
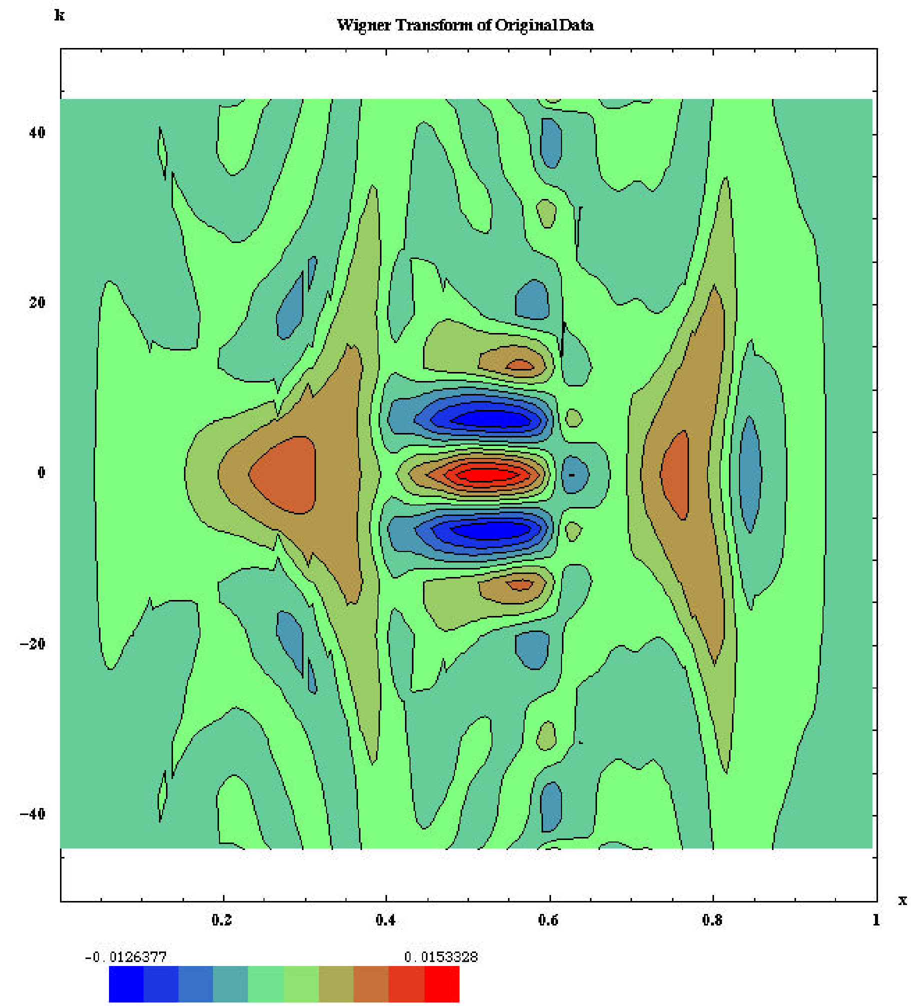




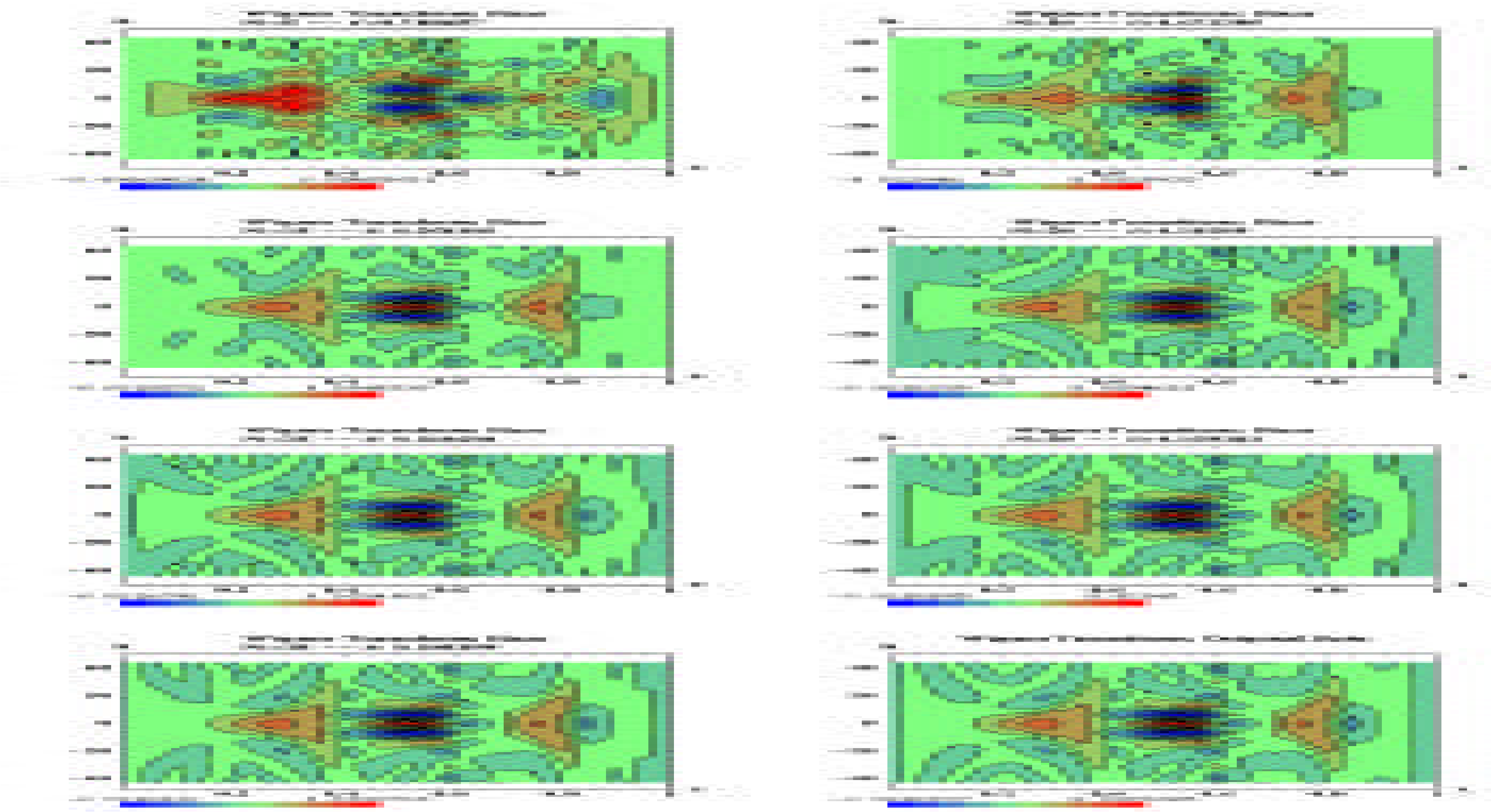
The Wigner Transform of the Random Shock Solution





Haar MRD in Wigner Phase Space

(5/79%, 10/47%, 15/26%, 20/20%,25/16%, 30/13%, 35/11%)



Daub3 MRD in Wigner Phase Space

(5/46%, 10/20%, 15/12%, 20/7%,25/4%, 30/3%, 35/2%)

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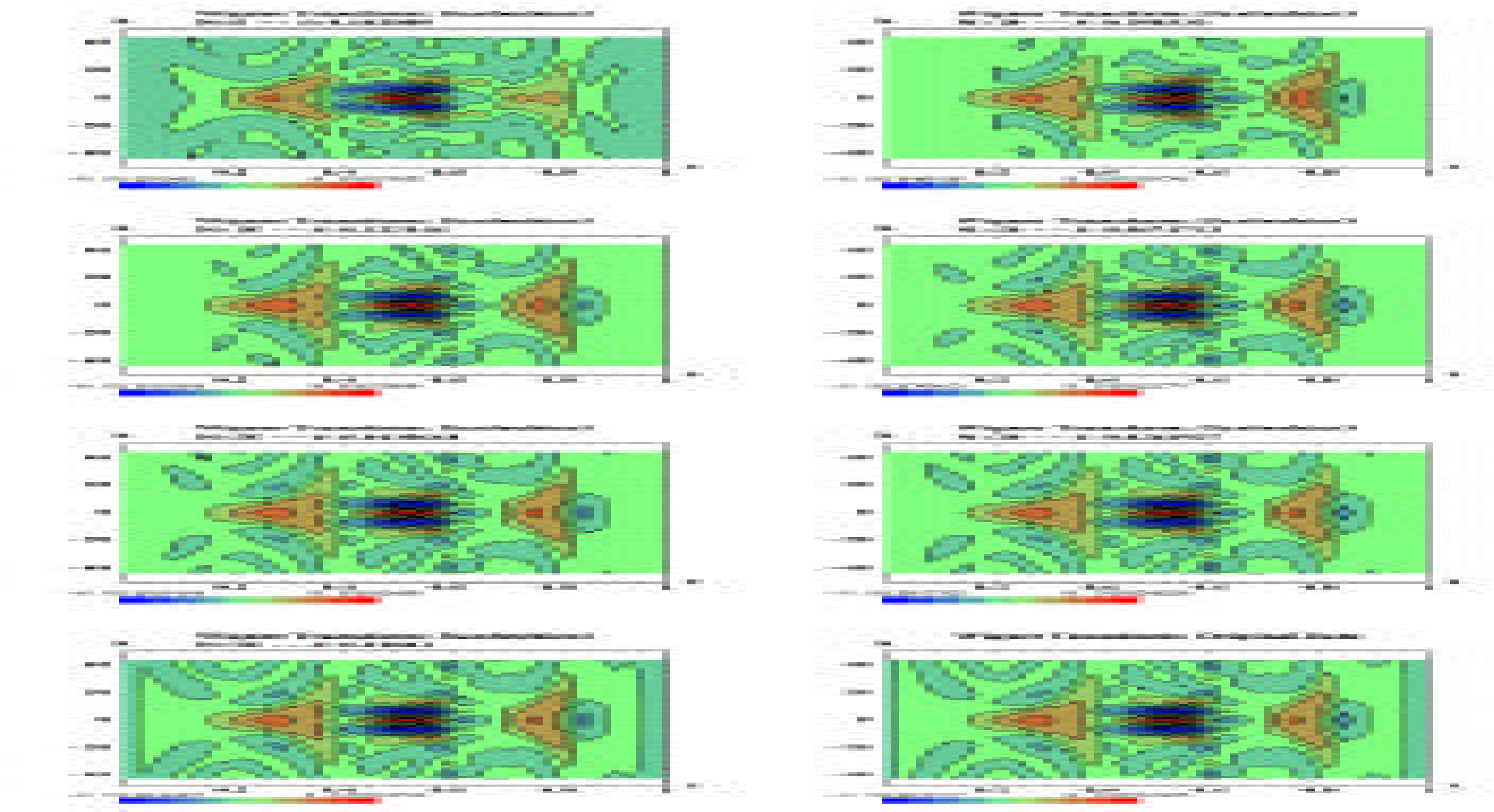

Successively Larger #s of Largest Daub3 Coefficients Used in the Reconstruction of $W_f(x,k)$

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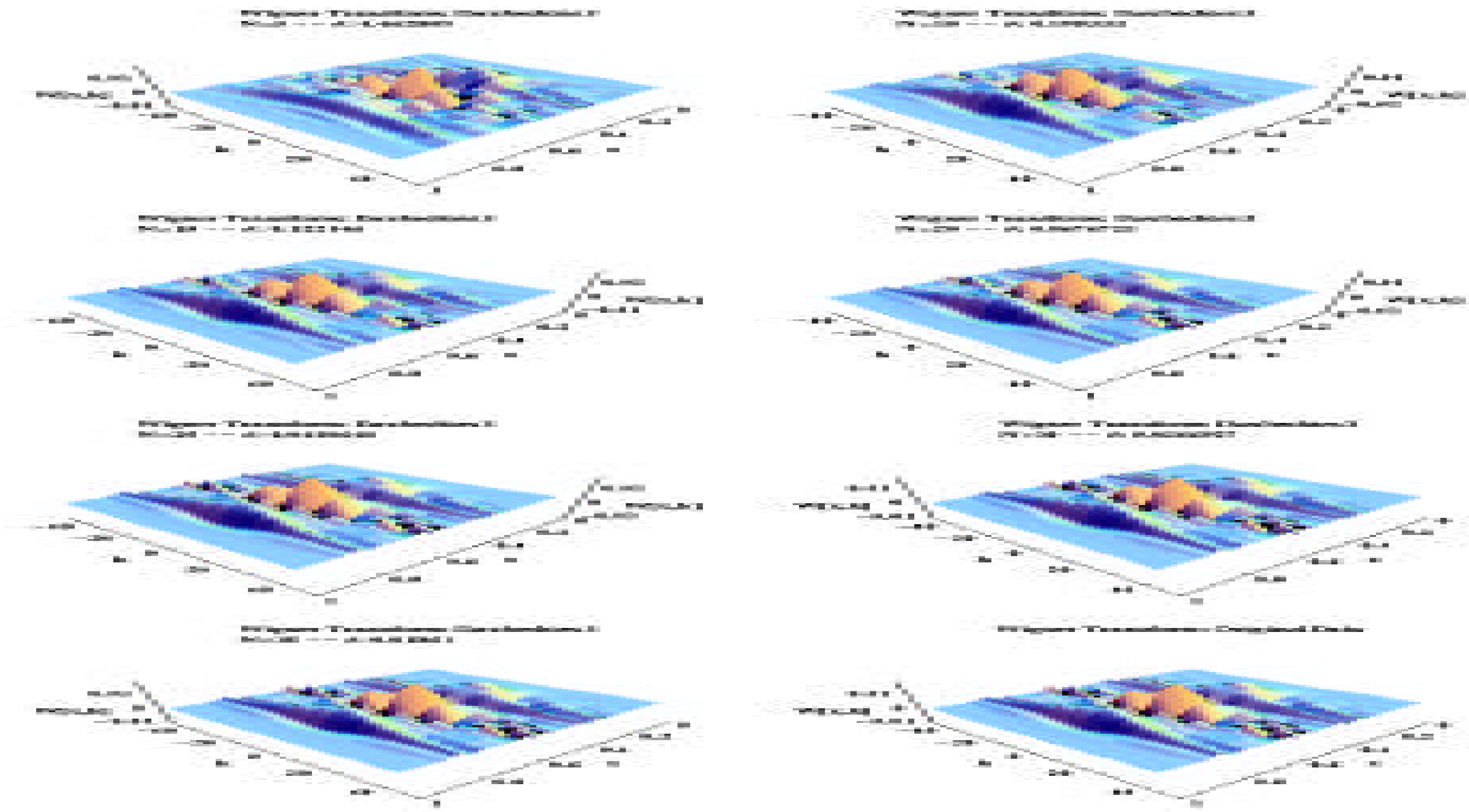

Reconstruction of $W_f(x,k)$ Using 5 to 35 Largest Haar WLTs in Multiples of 5

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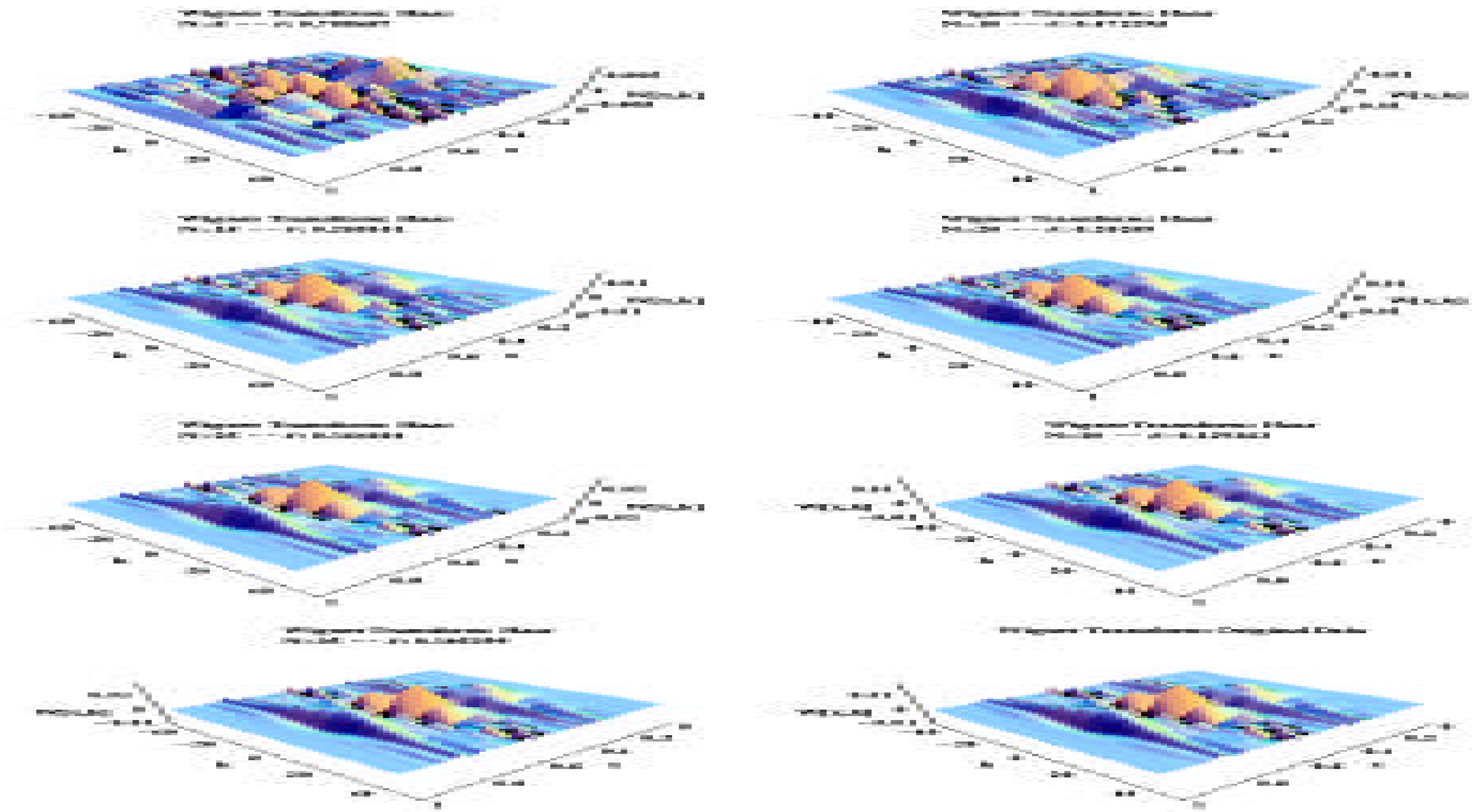

Quantitative Measures of the Relative Performance of Wavelet Families in WRMR Analysis

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- We can define metrics in phase space by which to automate the search through wavelet libraries for the optimum wavelet representation.
- Two choices are (i) Lebesgue type measures with local patches or weighting functions or (ii) the averaged cross-correlation function between the approximate signal's Wigner function and the actual signal's, normalized to the averaged auto-correlation function of the signal's Wigner function.

$$(i)\quad \mu_{WRMR}^{Lebesgue}(p,N,J) = \min\left(\sum_{patches} \frac{\left[\iint \left| \left(W_f - W_{f_{approx}} \right) w_{patch} \right|^p dx\, dk \right]^{1/p}}{\left[\iint \left| W_f \, w_{patch} \right|^p dx\, dk \right]^{1/p}} \right)$$

$$(ii)\quad \mu_{WRMR}^{Correlation}(N,J) = \min\left(1 - \sum_{patches} \frac{\left[\iint w_{patch} \left[\iint W_{f_{approx}}\left(x+x',k+k'\right) W_f\left(x',k'\right) w_{patch}\, dx'\, dk' \right] dx\, dk \right]}{\left[\iint w_{patch} \left[\iint W_f\left(x+x',k+k'\right) W_f\left(x',k'\right) w_{patch}\, dx'\, dk' \right] dx\, dk \right]} \right)$$

WRMR: A PRI Prescription

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- The Combined application of Wigner function phase space techniques and those of Multi Resolution Decomposition with Wavelets allows us to detect patterns **across scales** which might not be as apparent otherwise.
- By analyzing the time evolution of a signal on different scales in space, we can detect changes in patterns that are random from those that are **systematic, coherent or resonant**.
- Specially constructed phase space functionals beyond Wigner (bilinear) ones can capture even more pertinent information such as signaling the onset of 3 wave interactions, inverse cascades, etc. These, combined with **MRA with the optimum wavelet family** for that signal help in revealing the underlying physics.